EXTENDED ESSAY

MATHEMATICS

Deducing the graph representing the catenary without using differential equations

"How can one find the Cartesian equation of a catenary by simplifying the curve as masses attached and uniformly distributed alongside a string?"

Word count: 3259

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A special thanks to my supervisor, Hilde Hatlevoll and Prof. Tom Lindstrøm

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1. Introduction

Consider a chain hanging freely from two fixed points in the presence of gravity. The shape taking place is known as a *catenary* and can be found in a variety of places including spider webs and telephone lines. In this essay, I have analysed and modelled the catenary as a graph using my own method, which, after searching literature, seems not to have been published earlier. The seemingly easy problem is both complicated and fascinating as it serves as another area in mathematics where Euler´s number mysteriously appears.

Finding the function of the catenary is regarded as a classical problem in the field of calculus. The catenary was mistakenly believed to be parabolic by Galileo Galilei and was not solved until 1691 by Jakob Bernoulli who used differential calculus[1]. The traditional solutions depend on a second order differential equation stemming from a mathematical description of the curve, but in this essay, I have taken a reductionist approach by simplifying the catenary into its fundamental form: a simplified model which I have coined "simplified catenary".

The branches of math employed in my method are mainly trigonometry, function composition and calculus which build upon the description of how the gravity and tension force defines the catenary's shape. The bedrock of my method is to simplify the catenary into an idealized string with points of mass uniformly distributed alongside it. This simplification allows the use of drawing free body diagrams which illustrate the forces acting on each point. This gives light to a crucial trigonometric equation which

[1] E. H. Lockwood. "*A book of curves*". London: Cambridge University Press, 1961

I have presumptuously named after myself. With this equation, one can employ function composition to derive the Cartesian equation of the catenary.

2. Simplified catenaries

The starting point of my solution is to simplify the catenary as masses attached to a string spaced equally from each other. The figure below represents an inelastic, flexible, infinitely thin and weightless string held up by two fixed points with 5 points of mass (in black) attached to it.

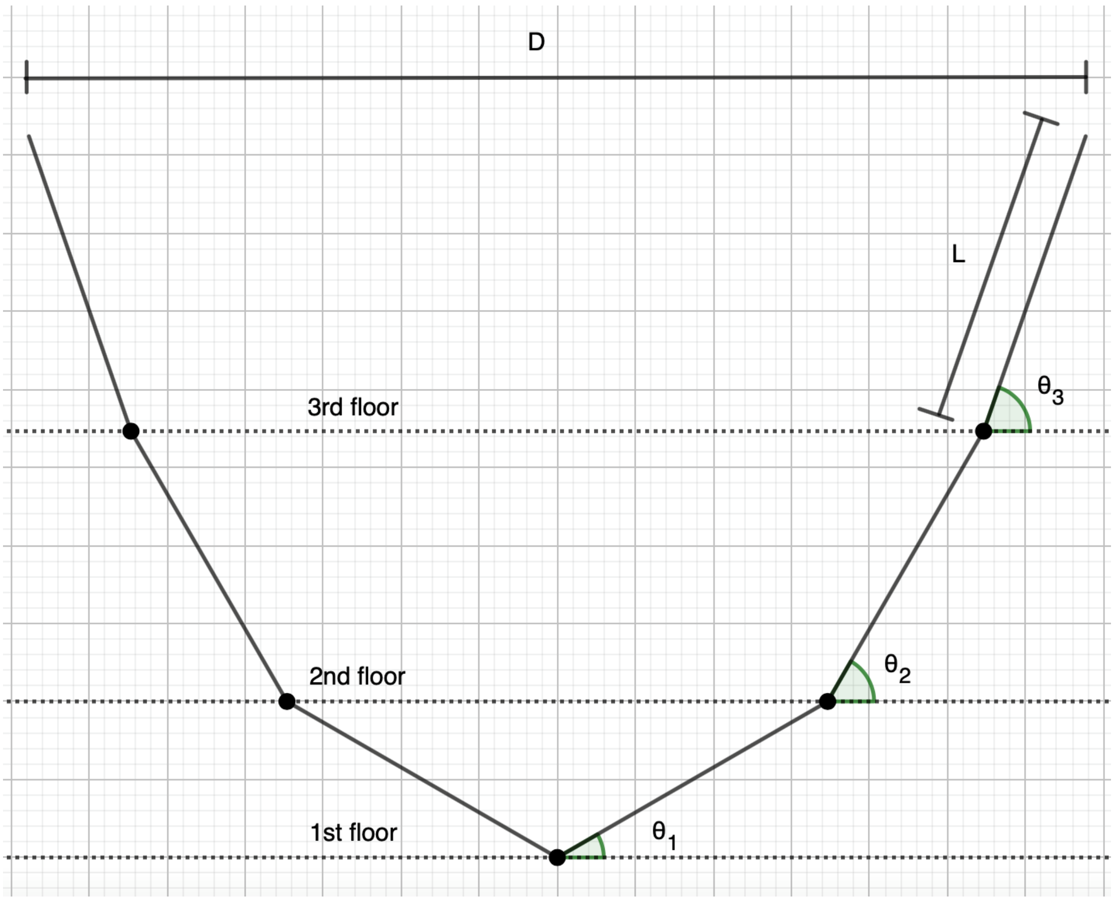

Figure 1: Third level catenary

The distances between each neighbouring mass along the string are identical and each segment is given an angle to the horizontal named θ_1, θ_2 and θ_3. In this essay, several simplified catenaries will be presented with a varying number of masses. All of them consist of one mass on the bottom and a varying number of pairs of masses above. To classify the catenary, I will look at the number of “floors” it consists of. The catenary illustrated above has 3 floors and we therefore classify it as a third level catenary. The first floor always consists of one mass.

As illustrated, the distance between the two fixed points supporting the masses will be referred to as D, and the length of each segment will be referred to as L.

By using common sense, we can make some assumptions of the simplified catenary, which are helpful in the mathematical analysis of its shape. As we can imagine, the catenary will be symmetrical as all the joints are equivalent in mass and equally distanced. More importantly, we can reason that each point of mass must be in mechanical equilibrium, meaning that the net force on each mass is zero.

2.1 First level catenary

The first level catenary is the simplest version as it only consists of one mass attached to the string. Its triangular shape may not closely resemble the curve but drawing a free body diagram of the point is the first step towards a powerful insight in the physics that governs the shape of a catenary.

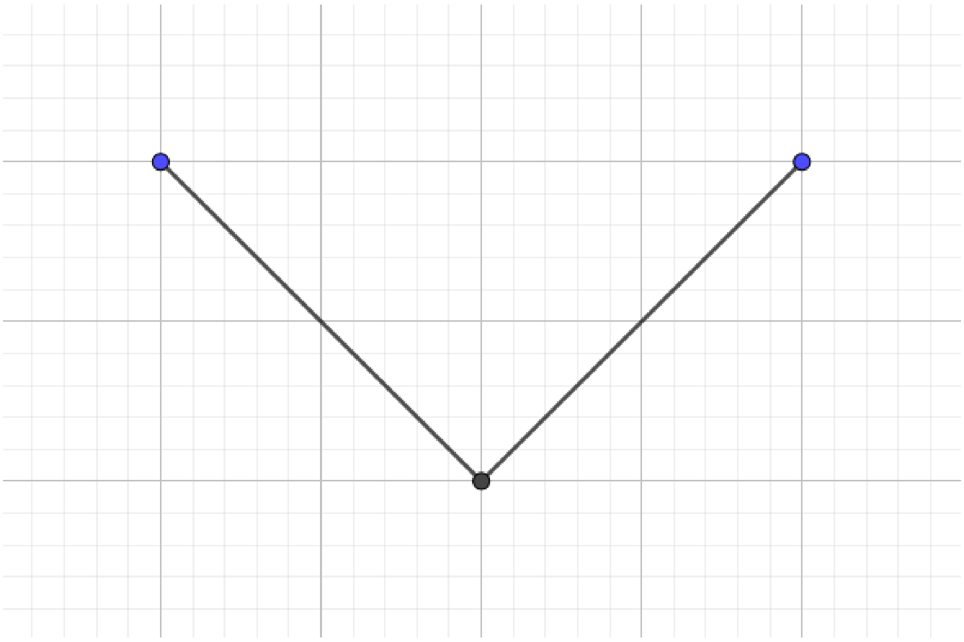

Figure 2: First level catenary

As mentioned, all the points of mass in the catenary are in equilibrium, meaning that the sum of all the forces acting on the points are equal to zero. *Figure 3* is a free body diagram of the forces exerted on the mass.

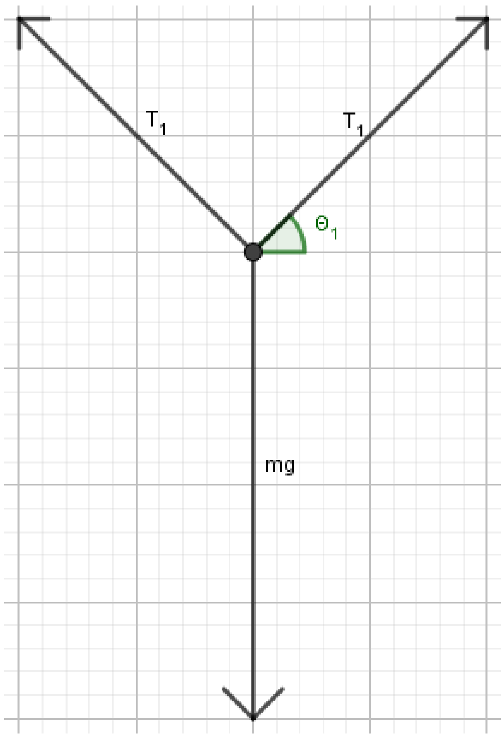

Figure 3: Free body diagram of the first level catenary

The mass is supported equally by the two segments resulting in tension. The sum of the vertical components of the tension equals the force mg.

By decomposing T_1 into its horizontal and vertical vectors, we obtain the following construction:

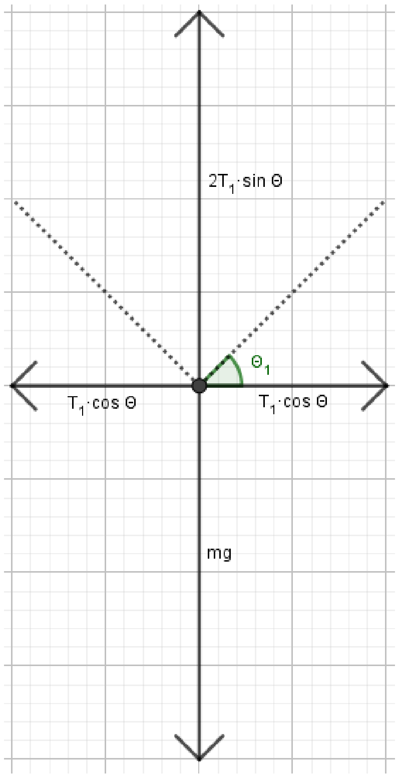

Figure 4: Decomposed vector forces of the first level catenary

As the mass is in equilibrium, the vertical vectors mg and $2 \cdot T_1 \cdot \sin\theta_1$ must be equal in magnitude. In other words, the horizontal vector component of T_1 equals $\frac{mg}{2}$:

$$2 \cdot T_1 \cdot \sin\theta_1 = mg$$

$$\therefore T_1 \cdot \sin\theta_1 = \frac{mg}{2}$$

We define T_1 in terms of mg and $\sin\theta_1$:

$$T_1 = \frac{mg}{2 \cdot \sin\theta_1}$$

One can make intuitive sense out of this expression: anybody who has held a heavy chain and stretched with both hands can imagine feeling the resisting force building as one pulls the ends further apart. In the same fashion, a decrease of $\sin\theta_1$ results in the tension growing fast. We will come back to this definition of T_1 later when analysing catenaries of higher levels.

Now that T_1 is defined in greater detail, we can express the horizontal component $T_1 \cdot \cos\theta_1$ as follows:

$$T_1 \cdot \cos\theta_1 = \frac{mg}{2 \cdot \sin\theta_1} \cdot \cos\theta_1$$

$$\therefore T_1 \cdot \cos\theta_1 = \frac{1}{2} \cdot mg \cdot \cot\theta_1$$

Now that we have defined the horizontal forces in greater detail, it is instructive to draw a new free body diagram.

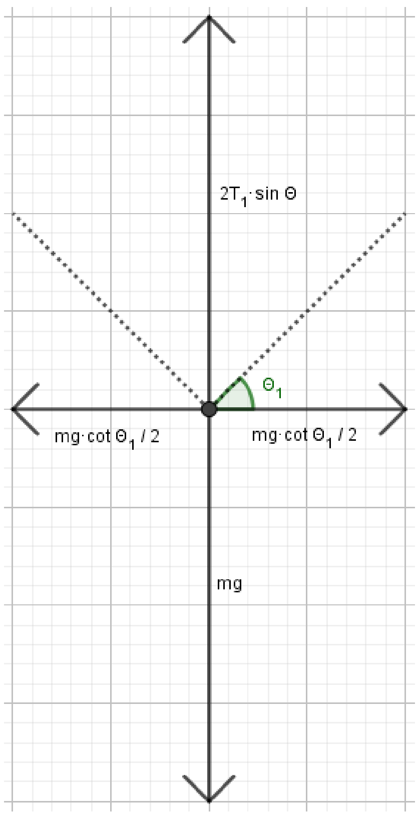

Figure 5: Decomposed vector forces of the first level catenary

2.2 Second Level Catenary

The second level catenary consists of two floors, making matters more complicated. Finding θ_1 and θ_2 in the second level catenary seems to be significantly harder because the shape can take many different forms.

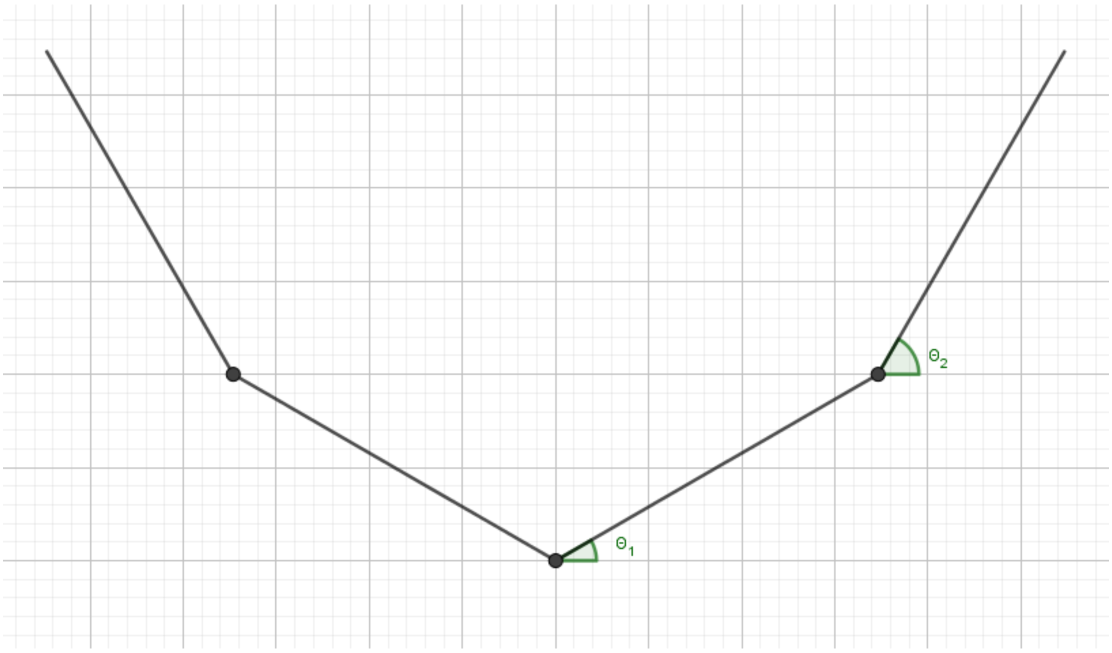

Figure 6: Second level catenary

By investigating the forces acting upon the new points however, we find that the force of gravity only allows for one possible shape of the catenary. We start by making a free body diagram of the upper right point in the second floor:

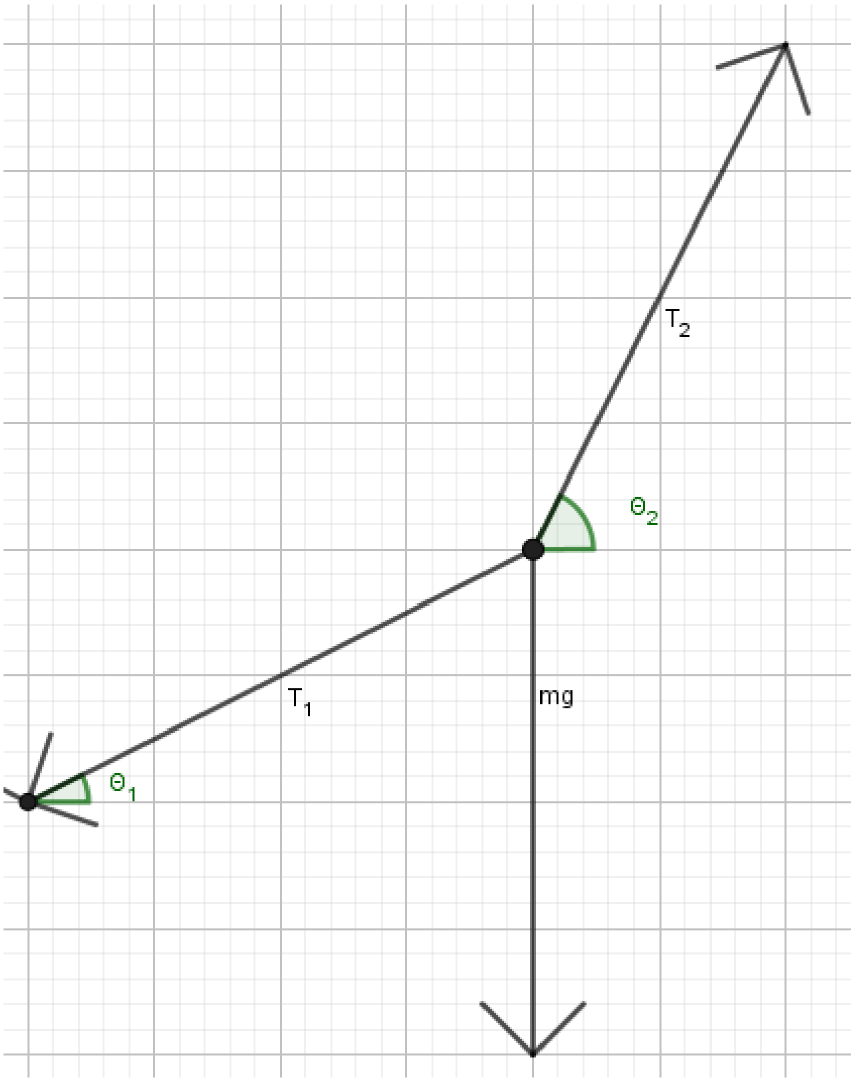

Figure 7: Free body diagram of the upper right point of a second level catenary

In *Figure 7*, T_2 represents the tension in the string of the second floor as a result of the three masses being pushed downwards by gravity. T_1, is the tension in the string of the first floor as a result of the bottom mass being pulled down by gravity as well. The force vector mg is simply the gravity acting on the mass in the second floor. The vectors in the free body diagram above are not drawn to scale, although they do cancel each other out.

By decomposing T_2 we get the following construction:

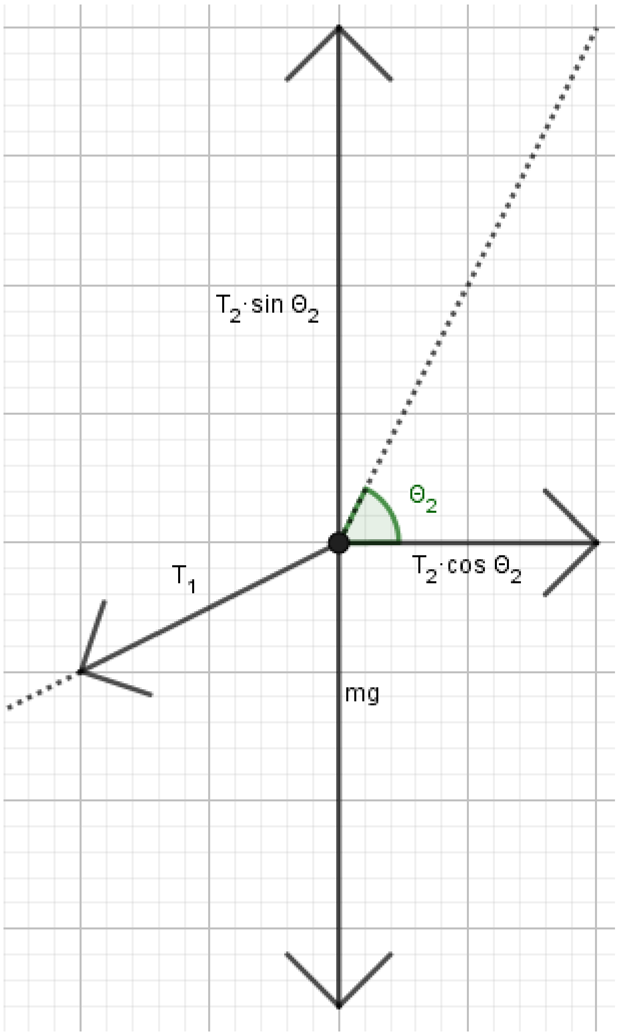

Figure 8: Decomposed vector forces of the upper right point

To express T_2 in greater detail we simply equate the vertical vector component $T_2 \cdot \sin\theta_2$ with mg and the vertical component of T_1, $T_1 \cdot \sin\theta_1$:

$$T_2 \cdot \sin\theta_2 = mg + T_1 \cdot \sin\theta_1$$

As shown in the previous section, the vertical component of T_1 is equal to $\frac{mg}{2}$:

$$T_1 \cdot \sin\theta_1 = \frac{mg}{2}$$

Thus, we can express the vertical component of T_2 in terms of mg:

$$T_2 \cdot \sin\theta_2 = \ mg + \frac{mg}{2}$$

$$\therefore T_2 = \ \frac{3 \cdot mg}{2 \cdot \sin\theta_2}$$

Now that T_2 is more specifically defined, we can redefine the horizontal component, $T_2 \cdot \cos\theta_2$, as follows:

$$T_2 \cdot \cos\theta_2 = \frac{3 \cdot mg}{2 \cdot \sin\theta_2} \cdot \cos\theta_2$$

$$\therefore T_2 \cdot \cos\theta_2 = \frac{3 \cdot mg}{2} \cdot \cot\theta_2$$

We redefine the vertical component, $T_2 \cdot \sin\theta_2$, as well:

$$T_2 \cdot \sin\theta_2 = \frac{3 \cdot mg}{2 \cdot \sin\theta_2} \cdot \sin\theta_2$$

$$\therefore T_2 \cdot \sin\theta_2 = \frac{3 \cdot mg}{2}$$

Now that the forces based on T_1 and T_2 have been defined more specifically we can draw a new free body diagram of the two points of a second level catenary:

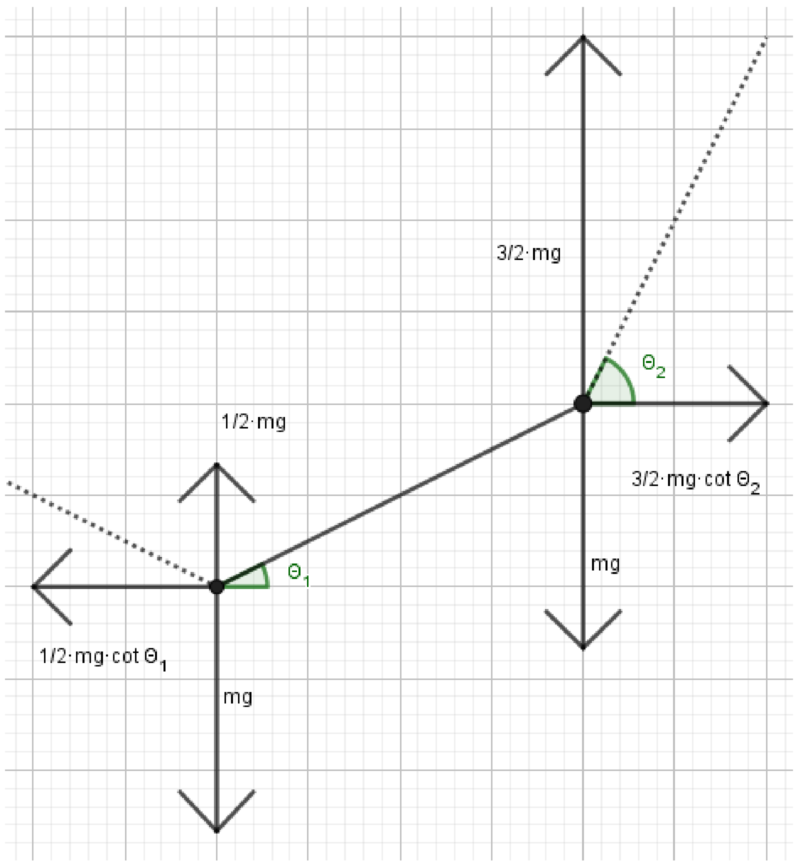

Figure 9: A free body diagram of two masses of the second level catenary

The illustration above is a free body diagram of the catenary as one body with two centres of mass. The forces constituting the horizontal component of the tension, T_1, are represented as $\frac{1}{2} \cdot mg \cdot \cot \theta_1$ and $\frac{3}{2} \cdot mg \cdot \cot \theta_2$.

We know the masses in a simplified catenary must be in perfect equilibrium. This fact is evidently consistent with regards to the vertical vector components:

$$\frac{1}{2}mg + \frac{3}{2}mg = mg + mg \Longrightarrow 2mg = 2mg$$

But the two horizontal components do not share the same obviosity. It is instructive to equate them:

$$\frac{mg \cdot \cot \theta_1}{2} = \frac{3 \cdot mg \cdot \cot \theta_2}{2}$$

$$\therefore \cot \theta_1 = 3 \cdot \cot \theta_2$$

$$\therefore \tan\theta_2 = 3 \cdot \tan\theta_1$$

The essence of this equation is surprising: it states that the slope of the second angle is three times greater than the slope of the first angle. This pleasing result elegantly dictates the value of $\tan\theta_2$ based on a given value for $\tan\theta_1$. We can find θ_2 in terms of θ_1 as follows:

$$\theta_2 = \arctan(3 \cdot \tan\theta_1)$$

2.3 Expressing angles in terms of width and length

This newly derived equation reveals the crucial relationship between θ_1 and θ_2 of the second level catenary:

$$\tan\theta_2 = 3 \cdot \tan\theta_1$$

But it would be helpful to deduce the two angles given a value for the segment length, L, and the width of the catenary, D.

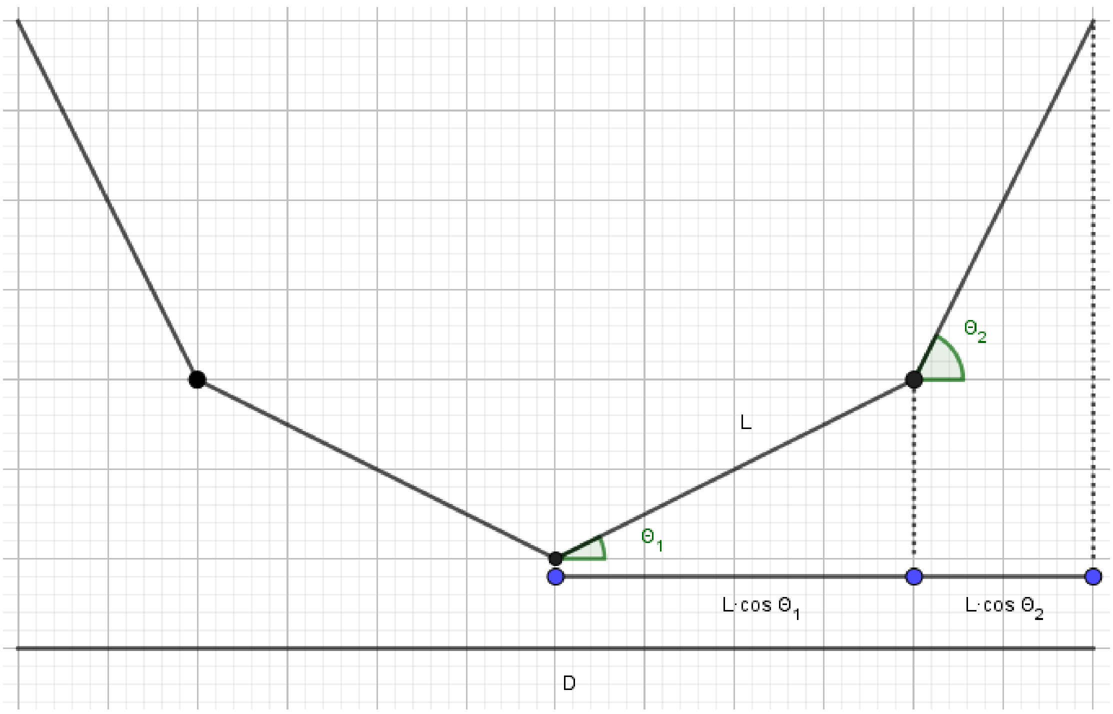

Figure 30: Second level catenary

With simple trigonometry we can express the width of the catenary as follows:

$$D = 2 \cdot (L \cdot \cos\theta_1 + L \cdot \cos\theta_2)$$

$$\therefore \frac{D}{2L} = \cos\theta_1 + \cos\theta_2$$

By applying the discovery of the relation between θ_1 and θ_2, we can obtain the following expression:

$$[\theta_2 = \arctan(3 \cdot \tan\theta_1)]$$

$$\therefore \frac{D}{2L} = \cos\theta_1 + \cos(\arctan(3 \cdot tan\ \theta_1))$$

It is seemingly too difficult to solve the equation for θ_1 for given values of D and L using classical algebra. Alternatively, one can solve it graphically, but this will not be covered in this essay.

3. Victor's Law

Victor's Law asserts that the slope of adjacent segments of a simplified catenary increases arithmetically. This section deals with proving this proposition.

The equation below was derived from equating the horizontal vector forces of two adjacent points of a second level catenary in section 2.2.

$$T_2 \cdot \cos\theta_2 = T_1 \cdot \cos\theta_1$$

$$\therefore \tan\theta_2 = 3 \cdot \tan\theta_1$$

It turns the seemingly difficult problem of finding one angle based on the other into a primitive task, but this relationship does not restrict itself to second level catenaries. Since all the points in a simplified catenary are in equilibrium, the horizontal vector forces of each mass must be equal to cancel each other out. In other words, we should be able to equate the horizontal vector component of any two masses in a catenary. As the formula for the horizontal component with respect to the floor, n, is $T_n \cdot \cos\theta_n$, we can obtain the following expression where k is another floor in the same catenary:

$$T_n \cdot \cos\theta_n = T_k \cdot \cos\theta_k$$

This is the first step towards deriving Victor's Law. By simplifying this expression, we can easily find the angle of any floor given a known value of the first angle.

Here are the equations for tension in the segment of each floor of the second level catenary which were derived in section 2.1 and 2.2:

$$T_1 = \frac{mg}{2 \cdot \sin\theta_1}$$

$$T_2 = \frac{3 \cdot mg}{2 \cdot \sin\theta_2}$$

With inductive reasoning, we observe that the tension of a segment with respect to the floor equals to half of the sum of the masses it supports multiplied by g and divided by $\sin\theta_n$, where θ_n is the angle of the string to the normal:

$$T_n = \frac{\sum mg}{2 \cdot \sin\theta_n}$$

The sum of the masses in a simplified catenary with respect to its level, n, can be expressed as an arithmetic sequence with a common difference of $2 \cdot mg$ and first term of mg. This should be intuitive as the first level catenary has one mass, and the second level catenary has one, plus two more. The third level catenary has one, plus two, plus two. Hence:

$$\sum mg = mg + (n-1) \cdot 2mg$$

$$\therefore \sum mg = (2n-1) \cdot mg$$

T_n can therefore be written like this:

$$T_n = \frac{(2n-1)\cdot mg}{2\cdot\sin\theta_n}$$

Thus, we can finally express the horizontal component, $T_n \cdot \cos\theta_n$, as follows:

$$T_n \cdot \cos\theta_n = \frac{(2n-1)\cdot mg}{2\cdot\sin\theta_n}\cdot\cos\theta_n$$

$$\therefore T_n \cdot \cos\theta_n = \frac{(2n-1)\cdot mg\cdot\cot\theta_n}{2}$$

Now we can continue simplifying the equation, $T_n \cdot \cos\theta_n = T_k \cdot \cos\theta_k$:

$$\frac{(2n-1)\cdot mg\cdot\cot\theta_n}{2} = \frac{(2k-1)\cdot mg\cdot\cot\theta_k}{2}$$

$$\therefore (2n-1)\cdot\cot\theta_n = (2k-1)\cdot\cot\theta_k$$

By algebraically manipulating this expression further we can express the tangent of any angle in terms of its floor and $\tan\theta_1$:

$$\therefore (2n-1)\cdot\cot\theta_n = (2\cdot(1)-1)\cdot\cot\theta_1$$

$$\therefore (2n-1)\cdot\cot\theta_n = \cot\theta_1$$

$$\therefore \tan\theta_n = (2n-1)\cdot\tan\theta_1$$

$$\therefore \tan\theta_n = \tan\theta_1 + 2\cdot\tan\theta_1\cdot(n-1)$$

As the equation reveals, the slopes of neighbouring segments increase arithmetically, where the first term is $\tan\theta_1$ and the common difference is $2\cdot\tan\theta_1$. This is Victor's Law and exposes the catenary as simplistic in nature.

In the following example, we will find the value of θ_4 given that θ_1 is 30° using the newly derived formula:

$$\tan\theta_n = (2n-1)\cdot\tan\theta_1$$

$$[n = 4\ and\ \theta_1 = 30°]$$

$$\therefore \tan\theta_4 = (2\cdot(4)-1)\cdot\tan 30°$$

$$\therefore \tan\theta_4 = 7\cdot\tan 30°$$

$$\therefore\ \theta_4 = \arctan\,(7 \cdot \tan 30°)$$

$$\boxed{\therefore\ \theta_4 \approx 76.1°}$$

3.1 Constructing simplified catenaries with Victor's Law

$$\tan\theta_n = (2n-1) \cdot \tan\theta_1$$

As we can see, the tangents of the consecutive angles in a catenary increase arithmetically with a common difference of $2 \cdot \tan\theta_1$. This makes it easy to calculate the angles of a simplified catenary and construct them. In the illustration below a fourth level catenary has been constructed using Victor's Law, where the tangent of the consecutive angles increases with a common difference of $2 \cdot \tan 30°$.

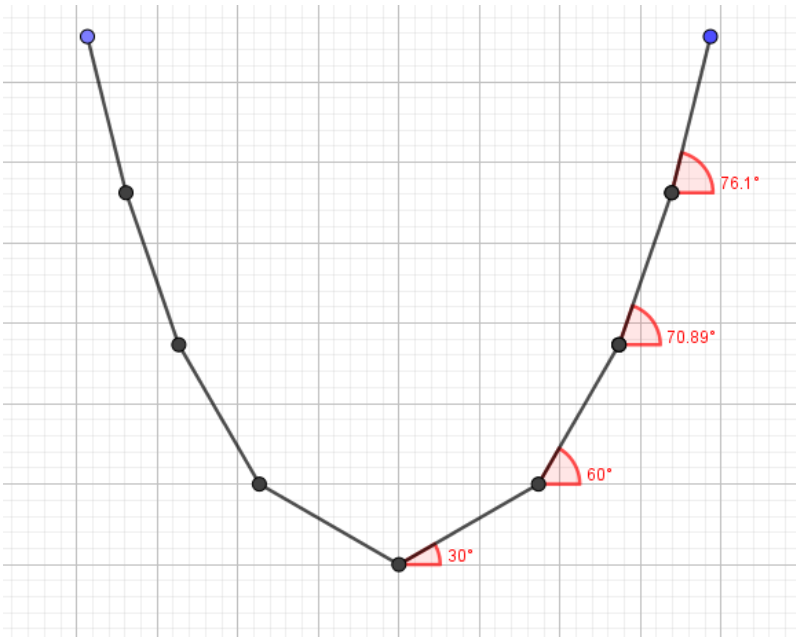

Figure 41: Fourth level catenary

The resemblance of a real catenary is clear.

4. The equation of the catenary

Victor´s Law makes it easy to construct simplified catenaries, but a fourth level catenary is still distant from a real one. This section is concerned with modelling the simplified catenary as a graph. One can imagine a real-life catenary as a chain of billions of atoms bonded together. Analogously to the simplified catenary, the width of each atom, L, is tiny and the slope of the line drawn between the bottom and adjacent atom is virtually zero.

The method I have come up with relies on the use of function composition and integration.

4.1 Composite method

The bedrock of this method is to express the width and height of a simplified catenary as functions with respect to the floor, n, followed by the composition of a new function based on the two. The function of the catenary, $f(x)$, can be expressed as follows

$$f(x) = H \circ W^{-1}(x)$$

, where H is a function of the height and W^{-1} is the inverse of the function of width. The following subsection is concerned with providing the mathematical justification for why this composite function yields the catenary.

4.2 Proof of Composition

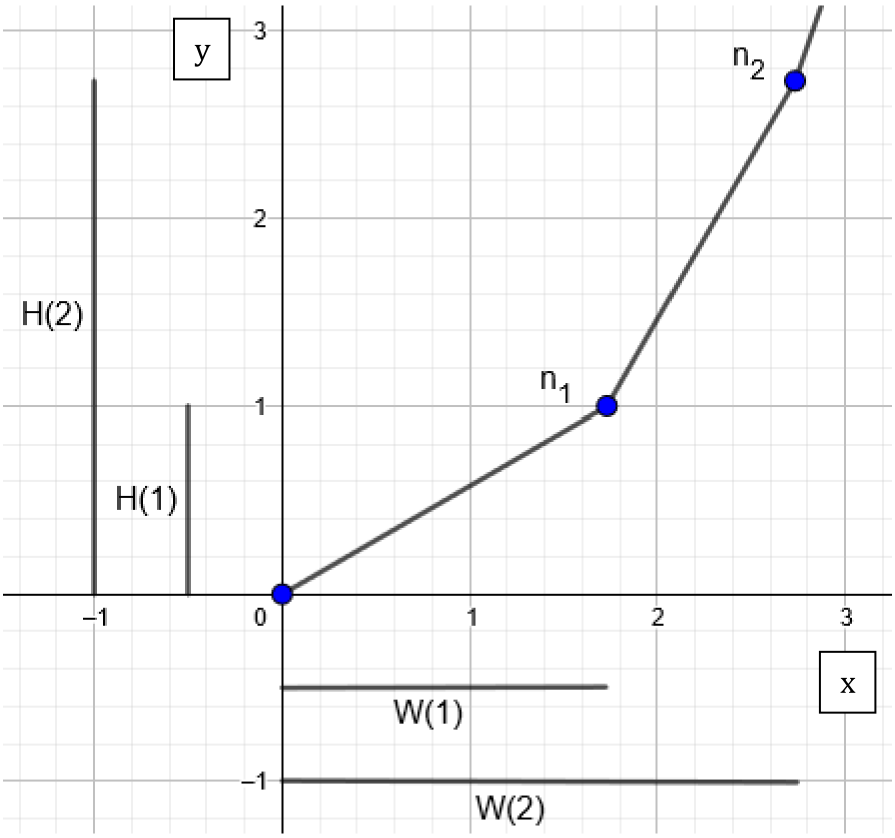

Figure 52: A simplified catenary as a function

The illustration above displays the right side of a simplified catenary. The position of the blue nodes represents the edges of the catenary of a given floor, n. The node, n_1, for example is the edge of the first level catenary. $H(1)$ and $W(1)$ give us the height and half of the width of the first level catenary. Similarly, $H(2)$ and $W(2)$ give us the height and half of the width of the second level catenary. The functions $H(n)$ and $W(n)$ give us the height and half of the width of simplified catenaries with respect to the floor, n.

The graph of the catenary, f, must map the variable $W(n)$ from its domain to $H(n)$ in its range. In other words, the input of our function, $W(n)$, yields $H(n)$ as the output. In summary, we can conclude the following operation of our function, f:

$$W(n) \xrightarrow{f} H(n)$$

$$\therefore f \circ W(n) = H(n)$$

The process of the function, f, is illustrated on this diagram:

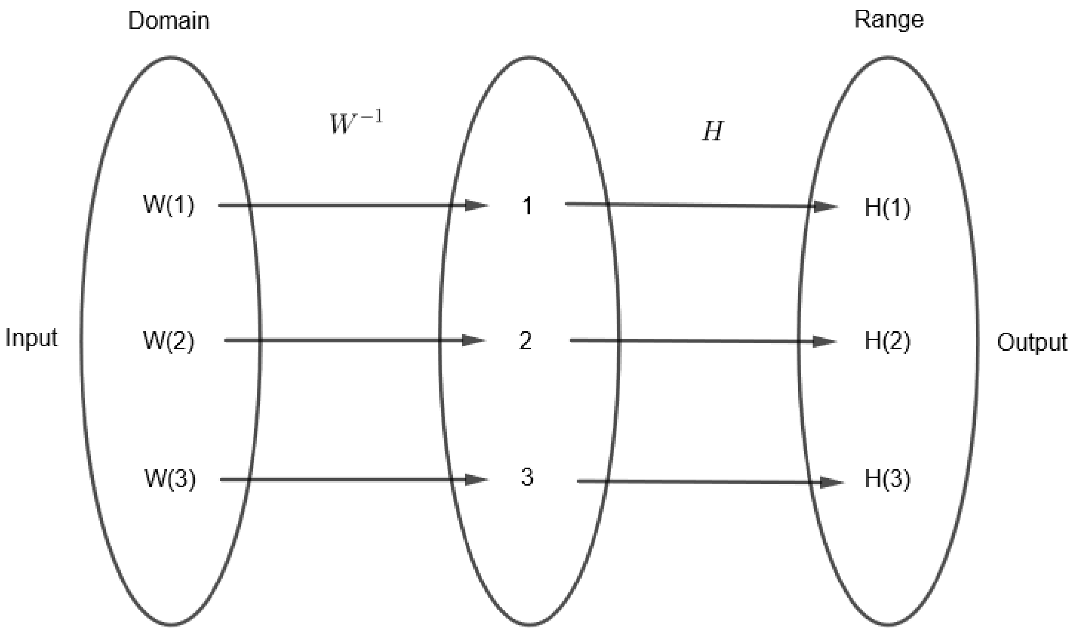

Figure 63: Diagram of the function, f

The inverse of W reverses W, yielding the domain, n, which is applied to H, resulting in $H(n)$. Hence, f must be $H \circ W^{-1}$. We test the proposition:

$$f \circ W(n) = H(n)$$

$$[f = H \circ W^{-1}]$$

$$\therefore H \circ W^{-1} \circ W(n) = H(n)$$

$$\underline{\therefore H(n) = H(n)}$$

Hence, we have the equation of the catenary:

$$f\big(W(n)\big) = H \circ W^{-1}(W(n))$$

$$[W(n) = x \; and \; f\big(W(n)\big) = y]$$

$$\therefore y = H \circ W^{-1}(x)$$

4.3 Defining W(n) and H(n)

As proved in section 2.3, the width of the second level catenary, D, can be expressed as follows:

$$D = 2L \cdot (\cos\theta_1 + \cos\theta_2)$$

To find the width of catenaries of higher levels, one simply must add more terms of $\cos\theta_n$. The width of a third level catenary for example is written like this:

$$D = 2L \cdot (\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$$

If one struggles to make intuitive sense out of this, it may help to look at *Figure 10*.

The general expression for the width of a catenary of the level, n, can be expressed as follows:

$$D(n) = 2L \cdot \sum_{i=1}^{n} \cos\theta_i$$

$$[\theta_i = \arctan((2i-1) \cdot \tan\theta_1)]$$

$$\therefore D(n) = 2L \cdot \sum_{i=1}^{n} \cos\arctan((2i-1) \cdot \tan\theta_1)$$

The function W however defines the width of one side of the catenary for a given n:

$$W(n) = \frac{D}{2}$$

$$\therefore W(n) = L \cdot \sum_{i=1}^{n} \cos\arctan((2i-1) \cdot \tan\theta_1)$$

The height of the catenary is expressed similarly to $W(n)$, except that cosine is replaced by sine:

$$H(n) = L \cdot \sum_{i=1}^{n} \sin \theta_i$$

$$[\theta_i = \arctan((2i - 1) \cdot \tan \theta_1)]$$

$$\therefore H(n) = L \cdot \sum_{i=1}^{n} \sin \arctan((2i - 1) \cdot \tan \theta_1)$$

The function $H(n)$ simply adds the height of all the floors beneath and including the nth floor.

4.4 Redefining W(n) and H(n) as definite integrals

The functions $W(n)$ and $H(n)$ have been defined, but in the form of series which are neither arithmetic nor geometric:

$$W(n) = L \cdot \sum_{i=1}^{n} \cos \arctan((2i - 1) \cdot \tan \theta_1) \quad \text{where } \mathbf{n} \in \mathbb{N}$$

$$H(n) = L \cdot \sum_{i=1}^{n} \sin \arctan((2i - 1) \cdot \tan \theta_1) \quad \text{where } \mathbf{n} \in \mathbb{N}$$

The inputs of these functions are also limited to natural numbers. Given this limitation and the fact that each are written as a series, it is difficult to find the inverse of $W(n)$ and apply it to $H(n)$. A possible solution is to approximate the series as definite integrals, opening the possibility for the functions to have a continuous domain rather than discrete.

4.4.1 W(n) as a definite integral

By representing the first two terms constituting the function $W(n)$ as rectangles, we can place them in order beneath the curve where the function is the summand of $W(n)$, which is $\cos\arctan((2i-1)\cdot\tan\theta_1)$. We obtain a Riemann sum:

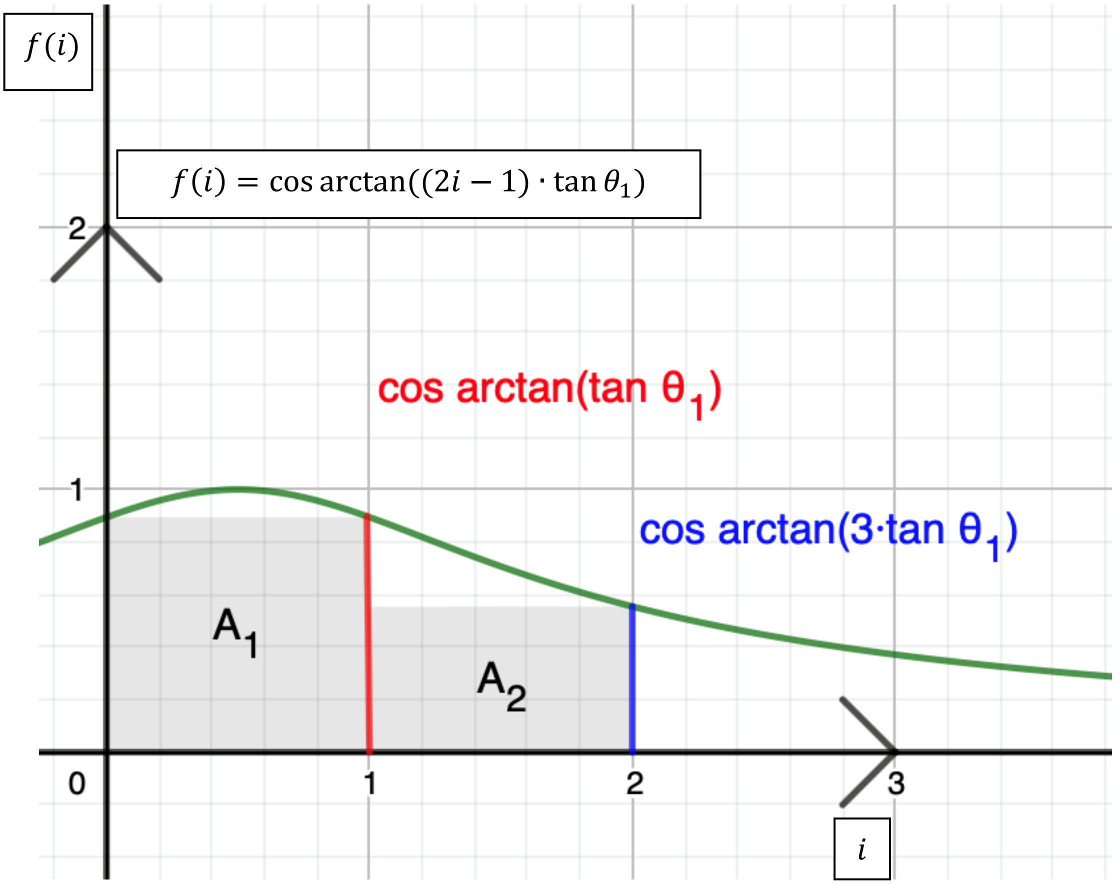

Figure 74: f(i) = cos(arctan((2*i*-1) $tan\,\theta_1$))

$$[tan\,\theta_1 = 0.5\ and\ L = 1]$$

The area of the first rectangle, A_1, equals the first term of $W(n)$ (in red). Likewise, A_2 equals the second term (in blue). We can approximate the sum of the first two terms as a definite integral:

$$W(2) = A_1 + A_2 \approx \int_0^2 \cos\arctan((2i-1)\cdot\tan\theta_1)\,di$$

We continue filling the bottom of the curve with rectangles based on the terms constituting $W(n)$:

Figure 85: $f(i)$ = cos(arctan((2i-1) $tan\ \theta_1$)))

$[tan\ \theta_1 = 0.5\ and\ L = 1]$

It is evident that the sum of the area of all the rectangles approximates to the total area under the curve, but not to a satisfactory degree.

We repeat filling rectangles beneath the curve, but this time with a smaller value for $\tan\theta_1$:

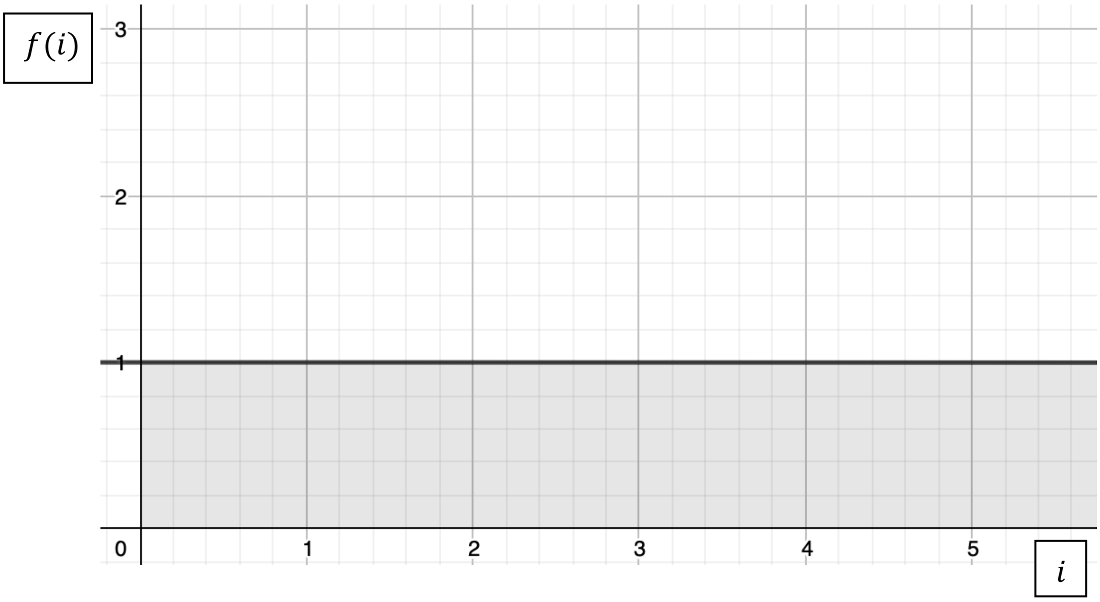

Figure 16: f(i) = cos(arctan((2*i*-1) *tan* θ_1)))

$$[tan\,\theta_1 = 0.005 \; and \; L = 1]$$

Interestingly, by lowering the value of $\tan\theta_1$, the slope flattens out, making the approximation method of filling rectangles beneath the curve highly efficient. Geometrically, the flattening of the curve makes intuitive sense as simplified catenaries with a small value for θ_1, have a smaller change in slope, and are therefore flatter. We can conclude the following:

$$\lim_{\tan\theta_1 \to 0} L \cdot \sum_{i=1}^{n} \cos\arctan((2i-1)\cdot\tan\theta_1) = \lim_{\tan\theta_1 \to 0} L \cdot \int_0^n \cos\arctan((2i-1)\cdot\tan\theta_1)\,di$$

In conclusion, for infinitesimal values of $\tan\theta_1$, we have the following expression for $W(n)$:

$$W(n) = L \cdot \int_0^n \cos\arctan((2i-1)\cdot\tan\theta_1)\,di \quad \text{where } \mathbf{n} \in \mathbb{R}$$

From here on, we will imagine $\tan\theta_1$ and L as infinitesimal in value to mimic a real catenary, but we do not assume that they both approach zero proportionally. Also, for the sake of simplicity, I will substitute $\tan\theta_1$ with θ_1 since $\tan\theta_1 = \theta_1$ for infinitesimal values of θ_1. Therefore:

$$\lim_{\theta_1\to 0} W(n) = L \cdot \int_0^n \cos\arctan((2i-1)\cdot\theta_1)\, di$$

We use Wolfram Alpha's online integral calculator[2] to solve the integral:

$$L \cdot \int_0^n \cos\arctan\big((2i-1)\cdot\theta_1\big)\, di = \frac{L}{2\cdot\theta_1}\cdot\sinh^{-1}\big((2n-1)\cdot\theta_1\big) + \frac{L}{2\cdot\theta_1}\cdot\sinh^{-1}\theta_1 + \mathrm{C}$$

We proceed by simplifying the result. We label the ratio between L and θ_1 as r:

$$\frac{L}{\theta_1} = r$$

This will become a parameter for the equation of the catenary:

$$\therefore W(n) = \frac{r}{2}\cdot\sinh^{-1}\big((2n-1)\cdot\theta_1\big) + \frac{r}{2}\cdot\sinh^{-1}\theta_1 + \mathrm{C}$$

Further simplification is done by assuming C to be zero, and eliminating the constant, $\frac{r}{2}\cdot\sinh^{-1}\theta_1$, as it approaches zero:

$$\lim_{\theta_1\to 0}\frac{r}{2}\cdot\sinh^{-1}\theta_1 = 0$$

$$\therefore W(n) = \frac{r}{2}\cdot\sinh^{-1}\big((2n-1)\cdot\theta_1\big)$$

4.4.2 H(n) as a definite integral

Re-expressing $H(n)$ as an integral is justified by the same logic as with $W(n)$. By decreasing the value of $\tan\theta_1$, the curve of the function of the summand of $H(n)$ flattens out, making the sum of the area of the rectangles identical to the area under the curve. Here are two diagrams of the function $\sin\arctan((2i-1)\cdot\tan\theta_1)$, with two different values of $\tan\theta_1$:

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[2] https://www.wolframalpha.com/calculators/integral-calculator/

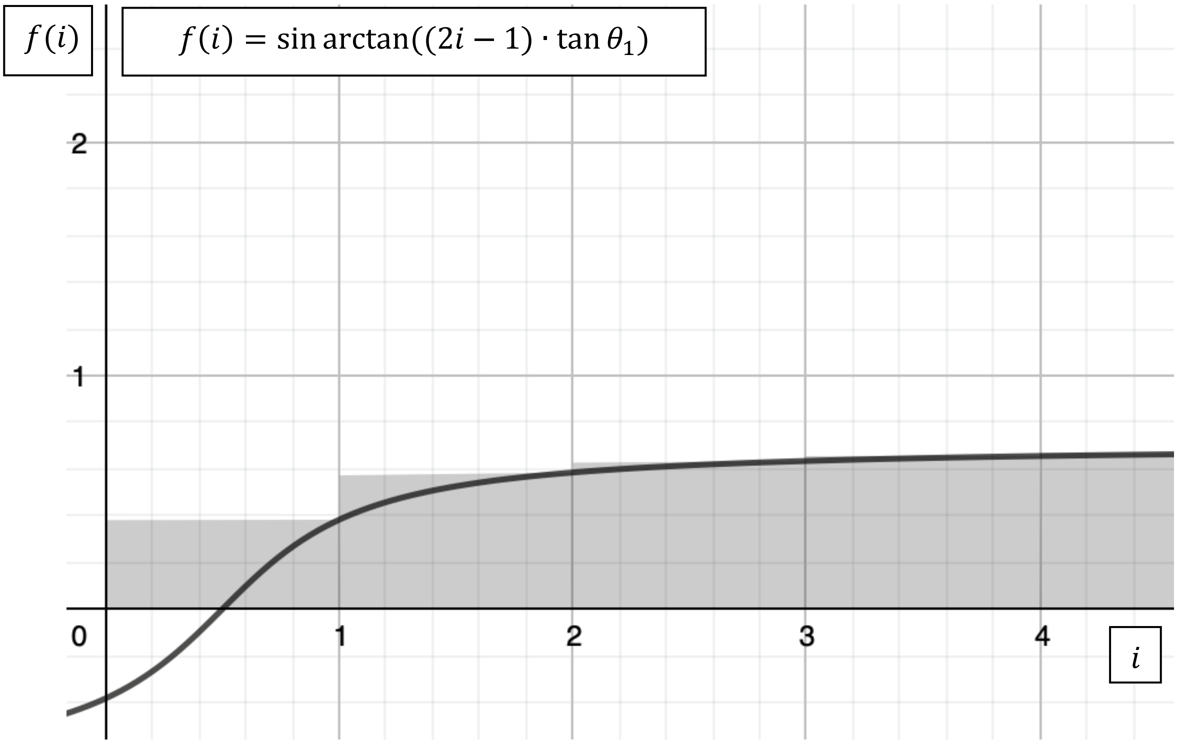

Figure 17: $f(i) = \sin(\arctan((2i\text{-}1)\ tan\ \theta_1))$

$[tan\ \theta_1 = 0.5\ and\ L = 1]$

Figure 98: $f(i) = \sin(\arctan((2i\text{-}1)\ tan\ \theta_1))$

$[tan\ \theta_1 = 0.005\ and\ L = 1]$

The curve is so flat that the rectangles are no longer visible. The important point however is that the shapes fill the space efficiently, due to the flat curve aligning almost perfectly with the horizontal sides of the rectangles.

Figure 18 is deceptive in the sense that it makes it look like the function approaches 0, while it does in fact approach 1 as i approaches infinity. This makes intuitive sense by imagining that the segments in the far infinite end of the catenary will eventually stand vertically. We can conclude with the following limit:

$$\lim_{\tan\theta_1 \to 0} L \cdot \sum_{i=1}^{n} \sin\arctan((2i-1)\cdot\tan\theta_1) = \lim_{\tan\theta_1 \to 0} L \cdot \int_0^n \sin\arctan((2i-1)\cdot\tan\theta_1)\, di$$

$$\therefore H(n) = L \cdot \int_0^n \sin\arctan((2i-1)\cdot\tan\theta_1)\, di \qquad \text{where } \mathbf{n} \in \mathbb{R}$$

We solve the definite integral using Wolfram Alpha's online integral calculator[2] once again to obtain our new expression for $H(n)$:

$$L \cdot \int_0^n \sin\arctan((2i-1)\cdot\tan\theta_1)\, di = \frac{L}{2\cdot\tan\theta_1}\cdot\sqrt{(2n-1)^2\cdot\tan^2\theta_1+1} - \frac{L}{2\cdot\tan\theta_1}\cdot\sqrt{\tan^2\theta_1+1} + \mathrm{C}$$

$$[\tan\theta_1 \to \theta_1 \; and \; \mathrm{C} = 0]$$

$$\therefore H(n) = \frac{L}{2\cdot\theta_1}\cdot\sqrt{(2n-1)^2\cdot{\theta_1}^2+1} - \frac{L}{2\cdot\theta_1}\cdot\sqrt{{\theta_1}^2+1}$$

$$[\frac{L}{\theta_1} = r]$$

$$\therefore H(n) = \frac{r}{2}\cdot\sqrt{(2n-1)^2\cdot{\theta_1}^2+1} - \frac{r}{2}\cdot\sqrt{{\theta_1}^2+1}$$

We can simplify the constant, $\frac{r}{2}\cdot\sqrt{{\theta_1}^2+1}$:

$$\lim_{\theta_1 \to 0} \frac{r}{2}\cdot\sqrt{{\theta_1}^2+1} = \frac{r}{2}$$

$$\therefore H(n) = \frac{r}{2}\cdot\sqrt{(2n-1)^2\cdot{\theta_1}^2+1} - \frac{r}{2}$$

4.5 Solving $\boldsymbol{H \circ W^{-1}(x)}$

Now that the functions have been rewritten such that they have continuous domains, we can resume solving the following composition:

$$f(x) = H \circ W^{-1}(x)$$

We find the inverse of $W(n)$ as follows:

$$W(n) = \frac{r}{2} \cdot \sinh^{-1}\big((2n - 1) \cdot \theta_1\big)$$

$$[W(n) \rightarrow x \textit{ and } n \rightarrow W^{-1}(x)]$$

$$\therefore x = \frac{r}{2} \cdot \sinh^{-1}\big((2 \cdot W^{-1}(x) - 1) \cdot \theta_1\big)$$

$$\therefore \frac{2}{r} \cdot x = \sinh^{-1}\big((2 \cdot W^{-1}(x) - 1) \cdot \theta_1\big)$$

$$\therefore \sinh(\frac{2}{r} \cdot x) = (2 \cdot W^{-1}(x) - 1) \cdot \theta_1$$

$$\therefore W^{-1}(x) = \frac{1}{2 \cdot \theta_1} \cdot \sinh \frac{2x}{r} + \frac{1}{2}$$

Finally, we apply $W^{-1}(x)$ to the domain of H:

$$f(x) = H \circ W^{-1}(x)$$

$$\therefore f(x) = \frac{r}{2} \cdot \sqrt{\left(2 \cdot (\frac{1}{2 \cdot \theta_1} \cdot \sinh \frac{2x}{r} + \frac{1}{2}) - 1\right)^2 \cdot {\theta_1}^2 + 1} - \frac{r}{2}$$

$$\therefore f(x) = \frac{r}{2} \cdot \sqrt{\left(\frac{1}{\theta_1} \cdot \sinh \frac{2x}{r}\right)^2 \cdot {\theta_1}^2 + 1} - \frac{r}{2}$$

$$\therefore f(x) = \frac{r}{2} \cdot \sqrt{\sinh^2 \frac{2x}{r} + 1} - \frac{r}{2}$$

Since the hyperbolic sine and cosine satisfy the following relation, we can simplify:

$$[\cosh^2 u = 1 + \sinh^2 u]^3$$

[3] http://math2.org/math/trig/hyperbolics.htm

$$\therefore \boxed{f(x) = \frac{r}{2} \cdot \cosh\frac{2x}{r} - \frac{r}{2}}$$

If we consider r to be 2, we arrive at the following cartesian equation of the catenary:

$$\underline{y = \cosh x - 1}$$

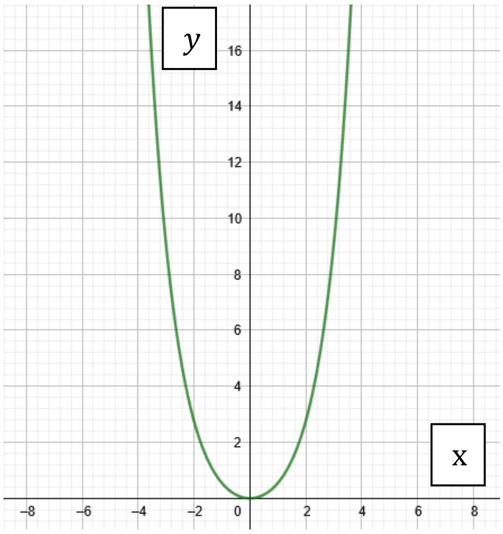

Figure 19: y = cosh(*x*)-1

$$[r = 2]$$

The equation can be written differently by substituting $\cosh x$ with its exponential form, revealing Euler's number:

$$[\cosh x = \frac{e^{x}+e^{-x}}{2}]$$ [3]

$$\therefore y = \frac{e^{x} + e^{-x}}{2} - 1$$

4.6 Comparing the equation with simplified catenaries

By recalling section 4.4.1, $\tan\theta_1$ was approximated as θ_1, and the parameter, r, equates to $\frac{L}{\theta_1}$. Hence, we can modify the equation of the catenary back into its un-simplified form:

$$y = \frac{L}{2 \cdot \tan\theta_1} \cdot \cosh\frac{2x \cdot \tan\theta_1}{L} - \frac{L}{2 \cdot \tan\theta_1}$$

Interestingly, the parameter gives us the ability to create catenaries which resemble simplified ones. Here is an example of a constructed second level catenary on a real catenary, with the same values for L and θ_1:

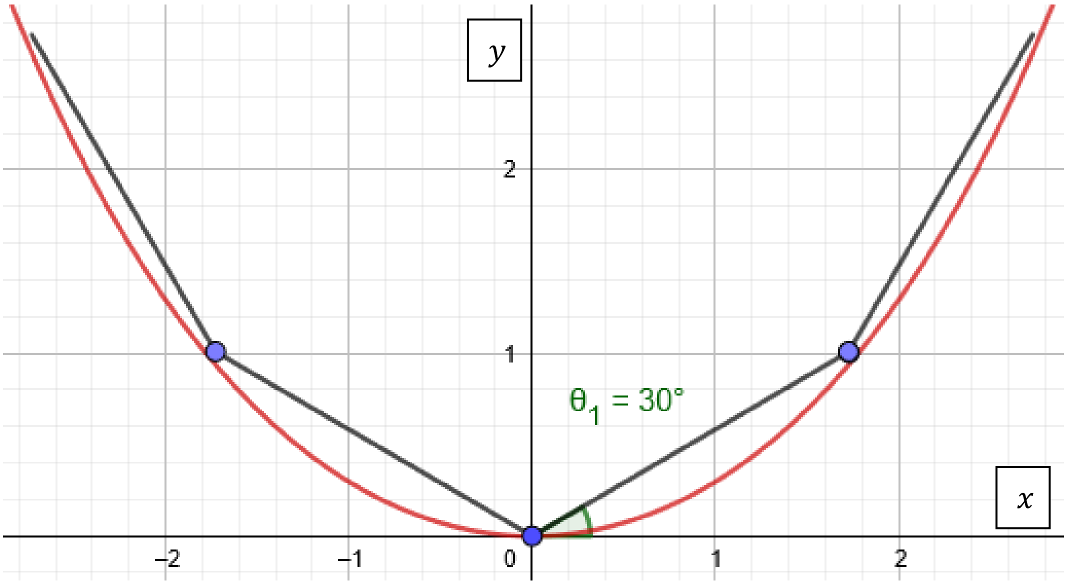

Figure 20: A second level catenary on a real catenary

$$[\theta_1 = 30° \; and \; L = 2]$$

If the slope of the line drawn through the bottom and adjacent atom and the average distance between the nuclei in a chain of atoms are known, one should, in theory, be able to replicate the curvature of the chain with high accuracy (assuming the chain of atoms to be inelastic).

5. Conclusion

This approach to deducing the shape of the catenary stands out from the traditional solutions as the math involved is less advanced. This allows for a wider range of people, including high school students, to comprehend the derivation of the equation of the catenary (with the help of an integral calculator). It also serves as an example of a relevant application of function composition: a subchapter in many high school math courses which seems to lack examples of practical applications.

Perhaps the most innovative aspect of this method is its reductionist interpretation of the catenary as masses on a string. It looks at the catenary in its essential form: atoms bonded together, forming a flexible chain. Analogously to the simplified catenary, the masses are atoms and the weightless string connecting the masses are chemical bonds. One could argue that this description of the catenary is more authentic than the traditional mathematical analysis which solely looks at the catenary as a curve. The mechanical flavour of this derivation also provides a new and insightful meaning to the parameter of the equation, r, as the ratio between L and $\tan\theta_1$. Additionally, it dictates the minima of the catenary to be positioned at origin in the face of any value for the ratio, r, which is a practical feature.

6. Bibliography

1) E. H. Lockwood. *A book of curves*. London: Cambridge University Press, 1961.

2) Wolfram Alpha, *Online Integral Calculator*.

https://www.wolframalpha.com/calculators/integral-calculator/

[Accessed 29th September 2019]

3) David Manura. *Hyperbolic Definitions.* Math2.org, 2005.

http://math2.org/math/trig/hyperbolics.htm

[Accessed 29th September 2019]

7. Appendix

Conventional method of finding the cartesian equation of a catenary:

https://www.youtube.com/watch?v=O2MCBzw6kVg