Water Rocket
Fuel Optimization

International Baccalaureate

Extended Essay

Year 2020

Contents

0.1 Introduction

0.1.1 Scope of Work

Rocket Science has been looked upon as a complex field of science with a huge potential for innovation by the academic community. Yet, one can study such a complex system by analyzing a simple, if not, ideal model using Newtons laws. It came to mind that investigating such a complex system, with the help of a simplified model, might prove an interesting study. This study is an attempt to understand, and perhaps recommend, a margin for how much fuel is needed to reach the optimal height of a rocket. Subsequently, an experiment was designed in which a model rocket was launched with varying amounts of water and compared to a simulation based on a simplified model. The first objective of this study is to develop an intuition as to how varying the water affects the height of the rocket. An extensive analysis of the uncertainties of the experimental setup was included as part of understanding how it influences the experiment. The second objective is to verify, if possible, the experiment with the theory and recommend a margin for which the height is optimized. Consequently an evaluation of the assumptions made in the model and the validity of conclusions made on this subject are included.

0.1.2 Research Question

How does variation in water influence the optimal height of a rocket before burnout?

0.2 Background Information and Literature

0.2.1 Conservation of Momentum

The momentum in a closed system is conserved. Using this fact one can arrive at the following conclusion in which v_e represents the exhaust velocity of the rocket

$$T = v_e \frac{dM}{dT}[2] \tag{1}$$

The change in mass can be applied by using the principle of conservation of mass. This allows us to express thrust in terms of pressure.(Refer to appendix A and D for a detailed derivation)

$$T = 2(P - P_a)A_e[3] \tag{2}$$

0.2.2 Conservation of Energy

In order to solve for thrust one needs to determine the exhaust velocity. Assuming in-compressible, non-viscous and irrotational flow one can use Bernoulli's equation in an attempt to find the exhaust velocity. This results in an equation relating the exhaust velocity to the change in static and dynamic pressure.

$$v_e = \sqrt{\frac{2(P - P_a)}{\rho_w}}[6] \tag{3}$$

A rigorous derivation of this has been presented in appendix C.

0.2.3 Work done by launcher Rod

A launcher Rod will be used in the experiment to launch the water rocket. Hence, the initial conditions can be determined by considering the work done by the launcher rod. Given the assumption that the gas behaves as an ideal gas one could arrive at the following relation

$$\frac{p_f}{p_0} = (\frac{V_0}{V_f})^{\gamma}[4] \tag{4}$$

Using this fact, and conservation of energy one can conclude that the total work, where k in $P_k V_k$ refer to the state of the gas.(Refer to appendix A for detailed derivation)

$$W = \frac{1}{\gamma - 1}[P_0V_0 - P_fV_f] - P_{atm}(V_f - V_0)[4] \tag{5}$$

0.2.4 Adiabatic Expansion

Equation 13 presents the thrust in terms of pressure. But the pressure in the rocket is still an unknown. Hence, an equation for pressure is needed.
The rocket launches, as it expels the water in which the compressed air expands considerably fast pushing out a significant amount of water from the bottle. Assuming that the air behaves as an ideal gas, a good approximation would be to assume that there was no heat exchange between the gas and its surroundings resulting in an adiabatic expansion.
Adiabatic expansion could be formulated as follows

$$PV^{\gamma} = K \tag{6}$$

In other words, this is the same as stating that

$$PV^{\gamma} = P_0 V_0^{\gamma} \tag{7}$$

where Po and Vo are the intial pressure and volume of air. Re-arranging equation 4 in terms of pressure gives

$$P = \frac{P_0 V_0^{\gamma}}{V^{\gamma}} \tag{8}$$

Differentiating both sides by time gives

$$\frac{dp}{dt} = -\gamma \frac{P_0 V_0^{\gamma}}{V^{1+\gamma}} \frac{dV}{dt} \tag{9}$$

Note that the change in volume is the exhaust velocity multiplied by its respective area(Refer to appendix D)

$$\frac{dp}{dt} = -\gamma \frac{P_0 V_0^{\gamma}}{V^{1+\gamma}} v_e A_e \tag{10}$$

Rearranging equation 8 for V gives

$$V = \frac{P_0 V_0^{\gamma}}{P}^{\frac{1}{\gamma}} \tag{11}$$

Substituting equation 11 into 12 gives

$$\frac{dp}{dt} = -\gamma \frac{P_0 V_0^{\gamma}}{\frac{P_0 V_0^{\gamma}}{P}^{\frac{1}{\gamma}1+\gamma}} v_e A_e \tag{12}$$

Substituting equation 3 for exhaust velocity gives

$$\frac{dp}{dt} = -\gamma \frac{P_0 V_0^{\gamma}}{\frac{P_0 V_0^{\gamma}}{P}^{\frac{1}{\gamma}1+\gamma}} \sqrt{\frac{2(P - P_a)}{\rho_w}} A_e \tag{13}$$

Equation 13 could now be integrated to find an expression for the pressure. However, this proves to be quite complex hence an iterative solution will be attempted.

0.2.5 Drag

To make the mathematical model realistic it is important to include drag.
The drag[1] force can be expressed as

$$D = \frac{1}{2} C_d A \rho_{air} v^2 [5] \tag{14}$$

where Cd is the drag coefficient, A is the area and v is the velocity. Note that the drag coefficient is independent of the size of the object and speed of airflow under subsonic speeds.
The drag coefficient, area and density can be encapsulated in a single constant allowing the simplification of equation 14 to

$$D = K * v^2 \tag{15}$$

0.2.6 Mass

Determining the mass of the rocket as a function of pressure is relatively simple.
Expressing mass in terms of volume might prove to be a good strategy

$$M = \rho V \tag{16}$$

The volume of the water decreases until all the water is ejected out of the nozzle. Hence the volume is simply the difference between the total volume, Vt, and the volume of air, V. Using equation 7 this could be written as

$$M = \rho_w (V_T - (\frac{P_0 V_0^\gamma}{P})^{\frac{1}{\gamma}}) \tag{17}$$

Adding the dry mass of the rocket, mr, leads to

$$M = \rho_w (V_T - (\frac{P_0 V_0^\gamma}{P})^{\frac{1}{\gamma}}) + m_r [3] \tag{18}$$

[1] https://www.grc.nasa.gov/WWW/k-12/airplane/drageq.html

0.3 Simulation

The simulation will be modelled using the equations derived from the **Information and Literature section**.

0.3.1 Iterative Solution

The first step is to consider the sum of forces acting on a rocket. Consider the diagram below.

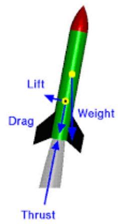

Figure 1: Adapted from waltonaero.wikidot.com

Given the coordinates in the diagram, the external forces on a rocket are

$$\sum F = T - D - W \tag{19}$$

Substituting equations 2, 15 for thrust and drag gives

$$\sum F = 2(P - P_a)A_e - K * v^2 - mg \tag{20}$$

Dividing by the mass gives an expression for the acceleration in terms of pressure and volume

$$\sum a = \frac{2(P - P_a)A_e - K * v^2}{M(P)} - g \tag{21}$$

The pressure can be determined iteratively using equation 13.

$$\frac{dp}{dt} = -\gamma \frac{P_0 V_0^\gamma}{\frac{P_0 V_0^\gamma}{P}^{\frac{1}{\gamma}1+\gamma}} \sqrt{\frac{2(P - P_a)}{\rho_w}} A_e \tag{22}$$

Multiplying both sides by change in time gives

$$dp = -\gamma \frac{P_0 V_0^\gamma}{\frac{P_0 V_0^\gamma}{P}^{\frac{1}{\gamma}1+\gamma}} \sqrt{\frac{2(P - P_a)}{\rho_w}} A_e dt \tag{23}$$

In other words equation 23 can be approximated as

$$P_{n+1} = p_n - \gamma \frac{P_0 V_0^\gamma}{\frac{P_0 V_0^\gamma}{P_n}^{\frac{1}{\gamma}1+\gamma}} \sqrt{\frac{2(P_n - P_a)}{\rho_w}} A_e \Delta t \tag{24}$$

Equation 24 allows iterative calculation for the pressure based on a time step delta t.

Using kinematic equations it is trivial to formulate that acceleration is change in velocity over time

$$a = \frac{\Delta v}{\Delta t} \tag{25}$$

It was found that acceleration is a function of pressure and velocity. Equation 25 is the same as claiming that

$$V_{n+1} = V_n + a(P_n, v_n)\Delta t \tag{26}$$

Equation 26 allows the calculation of velocity in an iterative fashion. Using the same procedure, it is trivial to find the height as a function of velocity.

$$y_{n+1} = y_n + v_n \Delta t \tag{27}$$

Using equations 21, 24, 26 and 27 it is possible to find the height of the rocket with small time increments.

0.3.2 Constants and initial conditions

There are several constants and initial conditions that needs attention before implementing the solution discussed in the previous subsection.

0.3.2.1 Precision

Eulers method of estimation is more accurate when the time step is quite small. This study uses a 4 decimal time step hence all the constants and values will be expressed in 4 decimal places.

0.3.2.2 Drag Constant K

To calculate the drag force acting on the rocket, the constant k, needs to be determined.
The constant, k , is equal to

$$k = \frac{1}{2} C_d A \rho_{air} \tag{28}$$

The drag of a water rocket has a small effect on the acceleration hence the drag coefficient, Cd, is not critical. But it is worth mentioning that the drag coefficient is independent of the size and speed of the airflow but is dependent upon the shape. Since the body is roughly ellipsoidal a value of 0.05 [2] is used for the body.

[2] Nasa provides an average of 0.05 for the drag coefficient

The fins has a major contribution to the drag hence a value of 0.1000 [3] is utilised. The area of the body is the cross-section normal to the airflow while the area of the fins,lateral area, is parallel to the airflow. The total area is thus the area of fins and the body. This was approximated to be around $2.9278 * 10^{-3}m^2$.
A value of $1.225kgm^{-3}$ was taken as the average for the density of air. Plugging the values in equation 41 results in

$$\mathbf{k = 2.689 * 10^{-4}kgm^{-1}} \tag{29}$$

The necessary constants have been summarised in the table below.

Constants	
Gamma(γ)	1.4000
Desnity(ρ_w)	$1000.0000kgm^{-3}$
Atmospheric Pressure(P_a)	$101325.0000Pa$
Time Step(dt)	$0.0001s$
Initial Pressure(P_0)	$551580.0000Pa$
Dry Mass(M_r)	0.4000kg
Cross-sectional area(A_e)	$1.963*10^{-5}\text{m}^2$
Drag Constant(k)	$2.689*10^{-4}\text{kgm}^{-1}$
Total Volume(V_T)	$0.0020m^3$
Gravitational Constant(g)	$9.8100ms^{-2}$

Table 1: A table of constants

0.3.3 Summary of Assumptions made in the Model

- Adiabatic Heat Transfer
- In-compressible Flow
- Ideal Gas Assumption
- No Energy Loss
- non viscous, irrational flow

[3]A nominal value for Drag was used

0.3.3.1 Initial Velocity

The next constant worth considering is the initial velocity.This is done by considering the total work done by the rod.However, the work done described by equation 5 requires the final state of pressure and volume. To tackle this, it was assumed that the final pressure is in equilibrium to the atmospheric pressure neglecting the residual pressure.

Since both the initial states of pressure and initial volume is known, the final volume was calculated using equation 11 to give

$$V_f = \frac{V_T - V_w}{0.5066^{\frac{1}{\gamma}}} \tag{30}$$

Using conservation of energy one might argue that

$$W = \Delta K + \Delta P \tag{31}$$

Since at the start the change in potential energy is almost 0,substituting equation 5 for work gives

$$\frac{1}{\gamma - 1}[P_0V_0 - P_fV_f] - P_{atm}(V_f - V_0) = \frac{1}{2}M_0v_0^2 \tag{32}$$

Solving equation 32 for initial velocity gives

$$v_0 = \sqrt{\frac{\frac{2}{\gamma-1}[P_0V_0 - P_fV_f] - 2P_{atm}(V_f - v_0)}{M_0}} \tag{33}$$

0.4 Design

The design of the simulation is based on the condition that the volume of water is above 0.

$$condition = V_T - (\frac{P_0V_0^{\gamma}}{P}) \tag{34}$$

The second term in equation 34 refers to the remaining amount of water in the rocket. The simulation is made such that when this condition breaks, maximum height is recorded.

0.4.0.1 Program

The Simulation[1][4] was implemented in java(Refer to appendix F for the source code) and the data points were exported to Graph Pad Prism[5]for analysis.

[4]This was programmed by Akash Amalan(BSc ComputerScience and Engineering) Github-Source:https://github.com/AJ730

[5]https://www.graphpad.com/scientific-software/prism/

0.5 Analysis and Graphical Representation

This section introduces the graphs derived from the simulation with the intention of uncovering the trends in the data points.

0.5.1 Results

The height was simulated for volumes of water as a fraction of the total volume ranging from 10% to 80%. Moreover, 0% and 100% were ignored as there is no pressure difference hence, no thrust. Similarly, 90% was also neglected as it was found that the thrust was not enough for lift off. The results found above were then plotted on a height versus time graph and presented in Appendix E.

0.5.2 Observations

It is not surprising that the results imply a one to one relationship between height and time. However, as the percentage of water increases the graph flattens out and in the most extreme case shows a parabolic curve. An instance from Appendix C was chosen and presented below.

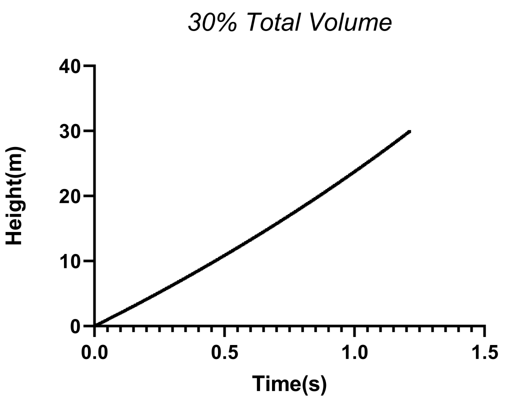

Figure 2: Water filled up to 30%

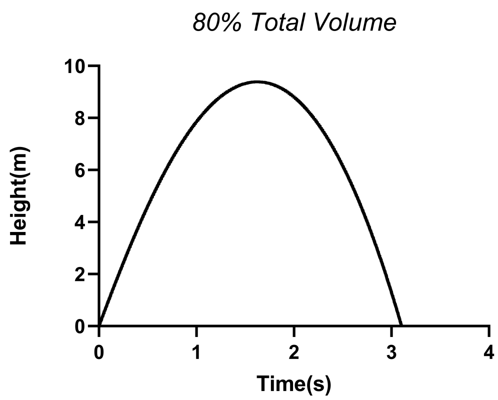

Figure 3: Water filled up to 80%

There is a drastic difference in shape between figure 5 and figure 6. Figure 5 seems intuitive as height is proportional to time, however figure 6 exhibits a parabola. The reason might be that not all water is used for the thrust resulting interestingly in a parabola.

It might also prove interesting to find a correlation, if it exists, between the mass and the maximum height. Consider the graph below.

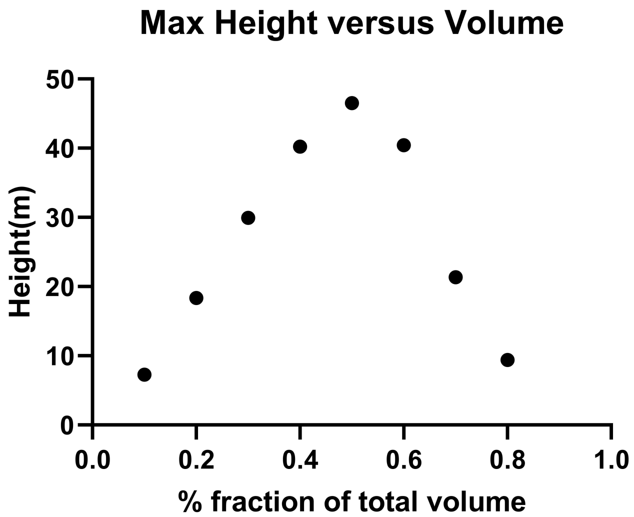

Figure 4: Graph illustrating optimum volume of water

The graph above, illustrates a parabola implying a quadratic relationship between the height and volume of water. It seems that the optimum amount of water is around 50%. However, in order to see how well it fits in a quadratic formula, an attempt to a quadratic regression was made using GraphPad.

This resulted in a R squared value of **0.9337** with a p test value of **0.6286**. The results seems to be well within the 95% confidence interval implying that a quadratic relationship is indeed a good fit. The results from the statistical analysis performed has been presented in the figure below

Goodness of Fit	
Degrees of Freedom	5
R squared	0.9337
Sum of Squares	103.0
Points above curve	4
Points below curve	5
Number of runs	5
P value (runs test)	0.6286
Deviation from Model	**Not Significant**

Table 2: Statistical Analysis for Quadratic fit

0.6 Experiment

0.6.1 Experimental Setup

An experiment is designed to study the motion of a rocket in an attempt to compare the theory with a real life experiment. To do so, different parts of a rocket was bought and assembled prior to the experiment.

0.6.1.1 Design Choice

A nominal plastic bottle with a volume of 2 litres was used as the body of the rocket. It was decided to cut the top of the bottle and replace it by pointy tip in an attempt to decrease drag at flight and increase mass at the tip for a stable flight. Similarly the bottom of the rocket was to be cut and replaced by a nose cone with an extremely small nozzle of 5mm at the end to increase the change in pressure and thus achieve greater height. Moreover, fins were to be attached at the bottom with the intention of stabilising the launch phase and the flight phase of the rocket.
A trigger system was required to control the pressure prior to launch. This was done by the inclusion of a mechanical switch system and a plug. It was decided that the rocket was pressurised by a bicycles pump with an embedded pressure gauge to read off the pressure conveniently. Consequently, a launchpad with a launch rod was used as a platform for stable launching.

0.6.1.2 Assembly

The different parts that was assembled to construct the water rocket have been portrayed in the images below.

Figure 5: Nozzle Overlay

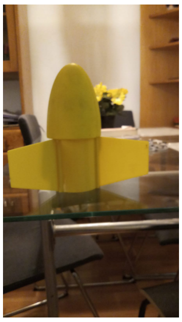

Figure 6: Fin and Top Overlay

Figure 7: Bicycle Pump

Figure 8: Launchpad

The Top was attached to the upper part of the bottle while the fins were attached to the bottom circumference of the bottle. The Nozzle overlay was attached to the bottom opening of the bottle and the assembled rocket was placed on the launchpad. The pump was then connected to the nozzle via the trigger. The trigger comprises of a plug which is pushed out mechanically when triggered. Once the rocket has been pressurised the plug falls in and seals the rocket. Once triggered the plug is mechanically pushed out and the rocket is free for launch. This assembly allows us to control the initial conditions such that the only dependent variable is the volume of water. The figures below illustrate the final assembly of the rocket.

Figure 9: Assembled Rocket

Figure 10: Digital Motion Camera

0.6.2 Measurement Description

The diameter of the bottle was determined by measuring the circumference of the bottle and a ruler was used for the measurement of the nozzle. The volume of the bottle was determined to be 2 litres by the product label and the dry mass was found by weighing the rocket on a kitchen scale.

0.6.2.1 Uncertainty in Measurement

The diameter is found by dividing the circumference by the constant PI. The uncertainty in circumference is mainly due to the parallax and random error. Hence, a value of $\pm 0.005m$ was assigned.
The relative uncertainty of the circumference is thus adapted as the uncertainty of the diameter.

$$D = 0.060m \pm \frac{0.005}{0.060} 100\% \tag{35}$$

The cross sectional area of the rocket is is thus expressed as

$$A = 0.00288 \pm 2\frac{\Delta D}{D} 100\% = 0.00288 \pm 2\frac{0.005}{0.06} 100\% \tag{36}$$

The diameter of the nozzle was measured to be 5mm using a caliper and the uncertainty was determined to be ± 0.0001 due to parallax error.

$$D = 0.005m \pm \frac{0.0001}{0.005} 100\% \tag{37}$$

The area can then be determined to be

$$A_e = 1.963 * 10^{-5} m + 2\frac{\Delta r}{r} = 1.963 * 10^{-5} m \pm \frac{0.0002}{0.005} 100\% \tag{38}$$

The dry mass was measured to be 0.4kg with an uncertainty of $\pm 0.001kg$. This is the same as

$$M = 0.400kg \pm \frac{0.001}{0.400} 100\% \tag{39}$$

0.6.2.2 Constants

Volume(m^3)	Mass(kg)	Diameter(m)	Area(m^2)	Diameter(Nozzle(m))	Area(Nozzle(m^2))
0.002	0.400	0.06	0.011	0.005	0.002

0.6.3 Instrument Description and Calibration

The instruments used to collect data include: ruler,thread,caliper, tracker(graphing Software)[6] and a 60 fps camera.

The camera, unfortunately, was not ideal due to the low frame rate. But to compensate for this, the experiment was recorded in slow motion. Nevertheless, it is important to consider how far the camera should be placed from the rocket to be able to record the whole flight. The field of view of the camera was read to be 65 degrees. After running several the simulation several times it was found that the maximum height of the rocket was in the range of 45 to 50 metres. This information can be used to find the minimum distance needed to film the experiment.

Consider the diagram below.

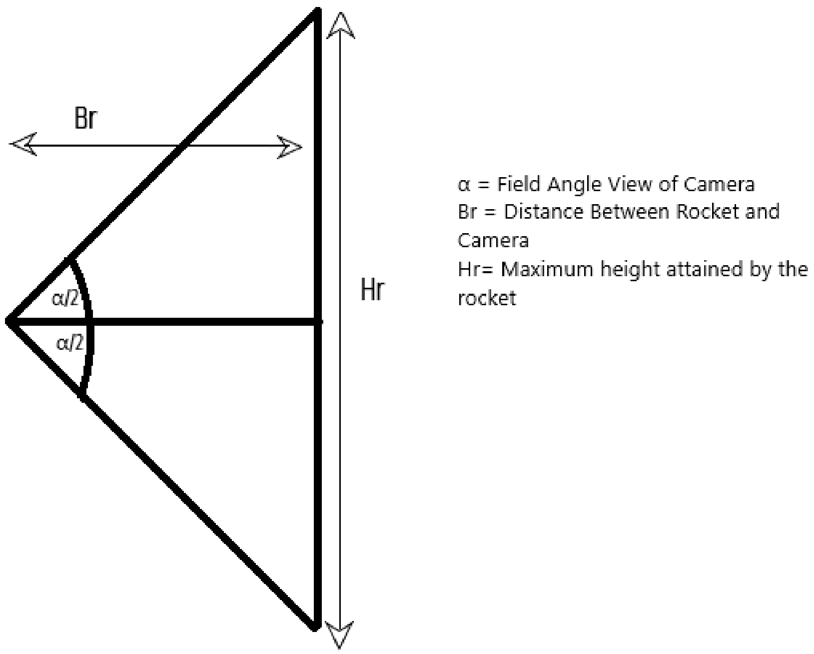

Figure 11: Determining minimum horizontal distance for video capture

The angle α was taken to be 65 degrees and the height Hr was taken to be 50 metres. Using the simple laws of trigonometry B_r can be expressed as a function of H_r and α.

$$B_r = \frac{1}{2}\frac{H_r}{\tan\frac{\alpha}{2}} = 39m \qquad (40)$$

Thus the camera was placed 39m from the launch position of the rocket. However, this would have repercussions as the quality of the video decreased drastically. Nevertheless, 39 meters is too far of a distance for a normal camera. But this study uses

[6] https://www.graphpad.com/scientific-software/prism/

a motion detecting camera, as previously showed in figure 13, which is able to zoom in with a magnitude of 20x. The camera was calibrated such that all rotations and movements parallel and normal to the z axis are restricted.

The graphical analysis program Tracker creates a point mass on the current position of the rocket and increments it over each frame. The point mass was chosen to be at the centre of the rocket due to the low quality of the camera. Despite, the drop in quality Tracker was able to track the rocket in each frame.

0.6.4 Experimental Procedure

The rocket was assembled as presented in figure 12. Additionally, a thread was connected to the rear of the rocket and the camera was set in position to record. A thread was added as it serves as a basis of comparison between the height calculated by tracker and that of the thread.
The experiment was conducted in a total of 8 trials. Each conducted twice in order to minimise the bias in the experimental data. Each trial was conducted by filling the bottle with water ranging from 10% of the total volume to 90% which was then followed by pressurising the rocket until an initial pressure of 551580Pa.
Once the initial pressure was reached, the camera was spontaneously set to record and the trigger was released. Once the rocket fell back to the ground the thread was measured and the camera was paused until the next trial.

0.6.5 Observations

It was observed that the rocket did not have a straight trajectory. One of the main reason for this might be the wind as the field was surrounded by trees. Another interesting observation was that all trials were within the 50m limit. This reinforces the model as it was theoretically determined that the maximum height would not exceed 50m.

0.6.6 Experimental Results

The following table summarises the experimental results for the height by measuring the thread with a tape.

Volume Ratio Amortised Height(m)	Initial Volume of Water(ml)	Measured Height 1(m)	Measured Height 2(m)
0.1	0.200 ± 0.005	4.00 ± 0.50	5.00 ± 0.50
0.2	9.000 ± 0.005	10.00 ± 0.50	11.00 ± 0.50
0.3	35.000 ± 0.005	36.00 ± 0.50	37.00 ± 0.50
0.4	39.000 ± 0.005	42.00 ± 0.50	44.00 ± 0.50
0.5	48.000 ± 0.005	49.00 ± 0.50	50.00 ± 0.50
0.6	40.220 ± 0.005	45.00 ± 0.50	47.00 ± 0.50
0.7	22.000 ± 0.005	23.00 ± 0.50	24.00 ± 0.50
0.8	9.380 ± 0.005	8.20 ± 0.50	9.00 ± 0.50

Table 3: Table representing the height recorded by measuring the thread

The height calculated through the software(Tracker) has also been depicted below. However, the error in the height introduced by taking the average between the data points in each trial. In other words, the error is dependent on the standard deviation between each set of points.

Volume Ratio	Initial Volume of Water(ml)	Calculated Height(m)
0.1	0.200 ± 0.005	4.20 ± 0.50
0.2	9.000 ± 0.005	10.05 ± 0.14
0.3	35.000 ± 0.005	36.28 ± 0.14
0.4	39.000 ± 0.005	45.23 ± 1.62
0.5	48.000 ± 0.005	50.56 ± 0.28
0.6	40.220 ± 0.005	48.012 ± 1.51
0.7	22.000 ± 0.005	23.55 ± 0.28
0.8	9.380 ± 0.005	8.35 ± 0.08

Table 4: Table representing the maximum height calculated by Tracker

0.6.6.1 Comparison of methods

It seems that the calculated height, with tracker, is within the error margin of the measured height.Table 3 has two data points hence an average height is needed.Therefore the heights have been amortised and presented in table 5.However, this study will use the calculated height, for consistency ,as the program was also used in other parts of this study.

Volume Ratio	Initial Volume of Water(ml)	Amortised Height(m)
0.1	0.200 ± 0.005	4.50 ± 1.00
0.2	9.000 ± 0.005	10.50 ± 1.00
0.3	35.000 ± 0.005	36.50 ± 1.00
0.4	39.000 ± 0.005	43.00 ± 1.00
0.5	48.000 ± 0.005	49.50 ± 1.00
0.6	40.220 ± 0.005	46.00 ± 1.00
0.7	22.000 ± 0.005	23.50 ± 1.00
0.8	9.380 ± 0.005	17.20 ± 1.00

Table 5: Table representing the amortised height recorded by measuring the thread

0.6.6.2 Graphical Representation and Comparison

A comparison of the results of the simulation and the theoretical results were made and presented in the graphs in Appendix G. A sample of the graphs have been presented here in order to derive some conclusions from the comparison between the simulation and experiment.

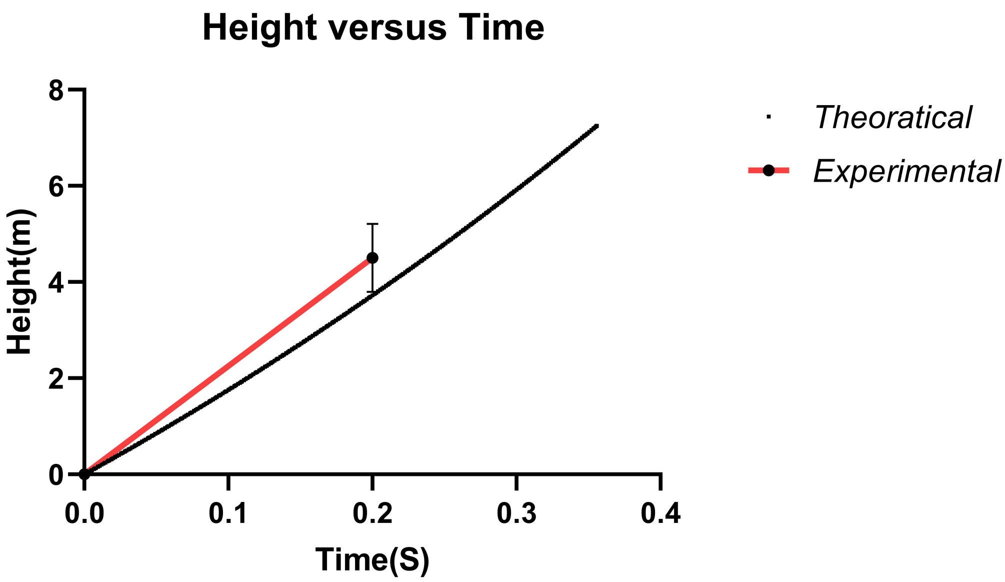

Figure 12: Height versus Time for a ratio of 10%

Figure 14 and Table 3 depicts that only 2 points was attained from the experiment as each time frame was 0.2 seconds and the flight was under 0.5 seconds. Given this premise, it is not possible to derive a valid conclusion. However, it does seem to follow the theoretical line within the error margin. Moreover, it seems at 10% only a

mere height of approximately 4m was reached implying that the water is too little to achieve even a decent height.
The second graph that might be of interest is that of 30% water since numerous scientific journals claim that 30% is the optimum volume of water to reach maximum height. Nevertheless, the model and the experiment has shown otherwise. But it is of interest to underline as to what degree the experimental agrees with the model.

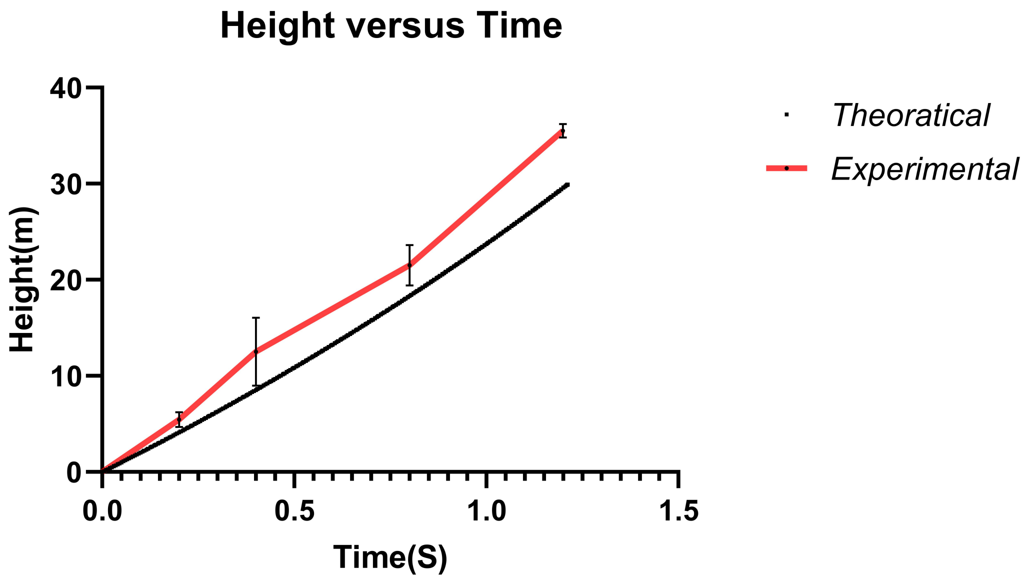

Figure 13: Height versus Time for a ratio of 30%

The graph above shows that the theoretical line is still within the error of the experimental. But it seems that the experimental height is slightly above the theoretical. Yet, it is not enough evidence to disprove the model.The software analysis is susceptible for the deviation as more often than not the point mass was tracked above the rocket than at the centre of the rocket.

The maximum height was recorded to be 49.50m at a volume ratio of 0.5. Referring back to figure 7, it is apparent that this agrees with the model. However, in order to conclude that the experiment indeed follows the model, one has consider the overall trend and the accuracy of the data points. Consider the graph below.

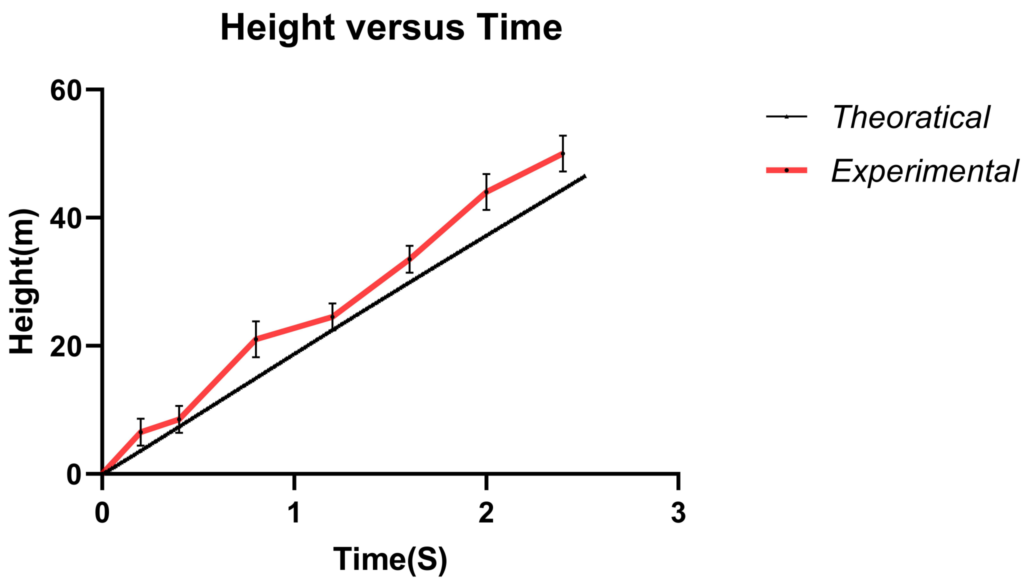

Figure 14: Height versus Time for a ratio of 50%

The graph above depicts that the general trend is linear. Nevertheless, only 3 out of 9 are within the margin of the theoretical trend line.Furthermore, it is also apparent that the experimental values tend to be higher than the theoretical values as in the previous graphs.
Given the above premise, it is hard to conclude that the experiment follows the model consistently. But it can be speculated that without the influence of wind and overestimation of point mass the experimental values would prove to be more accurate. On the other hand, it is quite evident that the variance in table 6, for a volume ratio of 0.5, is quite small suggesting that the points were rather precise.

It was also noticed in the simulation that the volume ratios 0.7 and 0.8 had a exceptional parabolic behaviour. It would be of uttermost interest to see if this behaviour is replicated in the experiment. Consider the graph below.

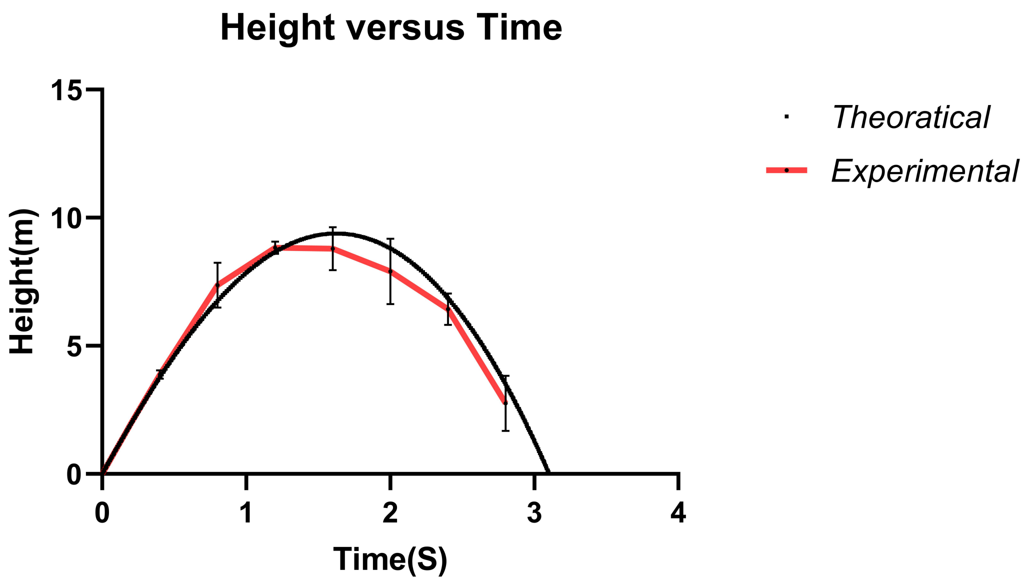

Figure 15: Height versus Time for a ratio of 80%

The experimental trend seems to adapt a parabolic behaviour as the simulation. All the points seem to be within the error margin of the experiment suggesting that the model is able to predict, rather accurately, the height gained by the rocket. Moreover, it was noticed that water was found in the rocket after its descent. This reinforces the conjecture that not all water was used for thrust implying that for most part of the flight the object was in free fall. Consequently, it must be also pointed out that there was no thrust at a volume ratio of 0.9 as predicted by the model.

0.7 Evaluation and conclusion

This investigation has led to rather interesting conclusions. First of all, we noticed that the graph for the optimal amount of water against height took interestingly a parabolic shape implying that there is a fine balance between the water and air that is needed for the highest range of the rocket. However, despite popular belief, the optimum amount of water was not found to be 30% but rather 50%. But this does not by any means conclude 30% as an unacceptable value but suggests that the thrust is dependent upon a lot of factors. Such factors comprise of nozzle diameter, volume and dry mass. In our case the rocket was small compared to the average rocket, with a limited volume of 2 litres and an extremely small nozzle diameter of 5 mm but in the case of a 6 litre volume it turns out that 30% is indeed the ideal amount for maximum height. Nevertheless, it is conclusive that the optimum amount for the rocket in test is 50% as shown both by experiment and simulation.

Secondly, we noticed that most of the experimental values were above the theoretical graph rather than below.However, the reason for this is still unknown, but speculation revolves around the fact that the drag was over-estimated when designing the simulation. Moreover, we also noticed that the simulation that the derivation for the initial velocity was not accurate mainly because of the assumption that there was no residual pressure. This could have been improved by using the fact that the final volume of gas should equal the volume of the bottle leading to a better approximation of the initial velocity. There are number of uncertainties in the experiment. Part of it was because of imprecise equipment used in measuring.Secondly the experiment should have been repeated more than 2 times such that the investigation turns out strong and more certain. Subsequently, it is undeniable that the wind had a drastic influence on the experiment. This could have easily avoided by performing the experiment indoors, if provided with the facility.

Quite a few questions emerge from this investigation. It would have been interesting to test different liquids or to repeat the experiment with different nozzle lengths. It would also be interesting to quantitatively measure the effect of drag in such a rigid body. We believe that this investigation has shed light on the fact that the optimum amount of water is dependent on the rocket itself and that there is no common value for all water rockets. However, we believe that the investigation highlights how to determine the maximum height using estimations such as Euler's method.

Word Count: 4000 words

Appendix A

Work Done by launcher Rod

[4] It is important to find the lift off initial condition as it will help in finding the initial velocity. Force applied on a closed area can be formulated as

$$F = \Delta PA \tag{A.1}$$

The launcher rod experiences a change in pressure over the cross-sectional area. The word over the length l can be described as

$$W = \int_0^l F_{launchRod} dz \tag{A.2}$$

Using equation A.2 it is possible to state that the force on the rod is the product of change in pressure and area

$$W = \int_0^l (P - P_{atm}) A dz \tag{A.3}$$

Using the fact that Adz corresponds to the volume. Equation A3 can be simplified to

$$W = \int_{v_0}^{V_f} (P - P_{atm}) dV \tag{A.4}$$

Expanding equation A4 gives

$$W = \int_{v_0}^{v_f} PdV - P_{atm}(V_f - V_0) \tag{A.5}$$

The gas expands from an initial volume,Vo, to a final volume,Vf. The work done in this interval can be formulated as

$$W = \int_{V0}^{V_f} Pdv \tag{A.6}$$

Substituting for P with equation A6 results in

$$W = P_0 V_0^{\gamma} \int_{V0}^{V_f} V^{-\gamma} dv \tag{A.7}$$

Integrating equation A7 yields

$$W = \frac{1}{\gamma - 1} P_0 [V_0 - V_f (\frac{V_0}{V_f})^{\gamma}] \tag{A.8}$$

Re-arranging equation 8 results in

$$\frac{p_f}{p_0} = (\frac{V_0}{V_f})^{\gamma} \tag{A.9}$$

Using equation A.9, equation A.8 can be simplified to

$$W = \frac{1}{\gamma - 1} [P_0 V_0 - P_f V_f] \tag{A.10}$$

Hence, in total, the work done by the launcher rod is given by

$$W = \frac{1}{\gamma - 1} [P_0 V_0 - P_f V_f] - P_{atm} (V_f - V_0) \tag{A.11}$$

Appendix B

Conservation of Momentum

[2] Consider a closed system as represented by the diagram below.

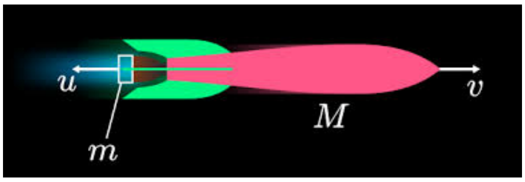

Figure B.1: Adapted from github.io/physics/rocket-equation

Figure 1 shows a closed system. Thus using the conservation of momentum [1] gives

$$(M + dm)v = M(v + dv) + dm(v - v_e) \tag{B.1}$$

This simplifies to

$$dm * v_e = Mdv \tag{B.2}$$

Dividing both sides by dt gives

$$M\frac{dv}{dt} = \frac{dm}{dt}v_e \tag{B.3}$$

Newtons second law states the following

$$F = ma \tag{B.4}$$

The propelling force of a rocket is generally referred to as thrust.Comparing equation 3 and 4 shows that the thrust, T, of the rocket is due to the ejection of water, M , over time and the exhaust velocity.

$$T = v_e\frac{dM}{dT} \tag{B.5}$$

[1] https://www.math24.net/rocket-motion/

Appendix C

Conservation of Energy

[6] It might be a challenge to measure the exhaust velocity. Assuming incompressible, non-viscous, irrotational flow Bernoulli's equations are applied along 2 points in the streamline in an attempt to relate equation 5 to pressure. Consider the diagram below.

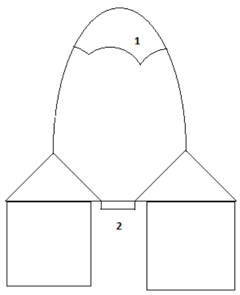

Figure C.1: Rocket Launch Position

Take an arbitrary point 1 at the surface of the water and a point 2 outside the nozzle. Pascal's principle suggests that the height between points 1 and 2 results in a pressure difference. However, since the height is considerably small the pressure difference is neglected. Moreover, the velocity at the surface is neglected in comparison to the velocity at the nozzle.

Applying Bernoulli's equation at the streamline between the two points gives

$$P_1 + \frac{1}{2}\rho_1 v_1^2 + \rho g_1 y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g_2 y_2 \tag{C.1}$$

Given the assumptions, equation 6 simplifies to

$$P_1 = P_2 + \frac{1}{2}\rho v_2^2 \tag{C.2}$$

A change of notation here is desired. Note that the Pressure at point 1 is the point 2 is the atmospheric pressure. Respectively the velocity at point 2 is the exhaust velocity and the density at point 2 is that of water.

$$P = P_a + \frac{1}{2}\rho_w v_e^2 \tag{C.3}$$

Equation 8 can now be solved for the exhaust velocity as a function of pressure. Rearranging equation 8 gives

$$v_e = \sqrt{\frac{2(P - P_a)}{\rho_w}} \tag{C.4}$$

Appendix D

Conservation of mass

[3] To determine the thrust it is also required to devise a way to find the mass flow. To begin with the mass flow can be realised by the product of volume flow and the density

$$\frac{dM}{dt} = \frac{dV}{dt}\rho_w \tag{D.1}$$

Consider the diagram below.

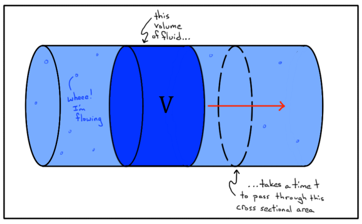

Figure D.1: picture adapted from Youtube Channel

Given a section of volume in a time frame one can argue that the change in volume is equivalent to the exhaust velocity times the area.

$$\frac{dM}{dt} = v_e A_e \rho_w \tag{D.2}$$

Combing equations 11 and 5 give

$$T = v_e^2 A_e \rho_w \tag{D.3}$$

Substituting 9 for exhaust velocity would result in

$$T = 2(P - P_a)A_e \tag{D.4}$$

Finding the thrust is thus determined by the pressure difference.

Appendix E

Graphs

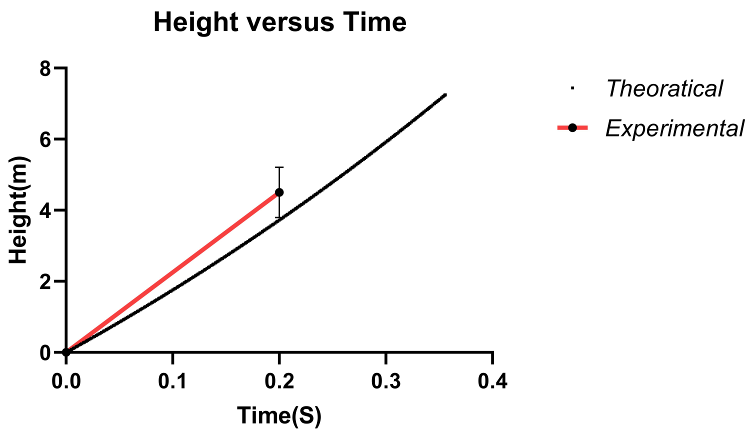

Figure E.1: Water filled up to 10%

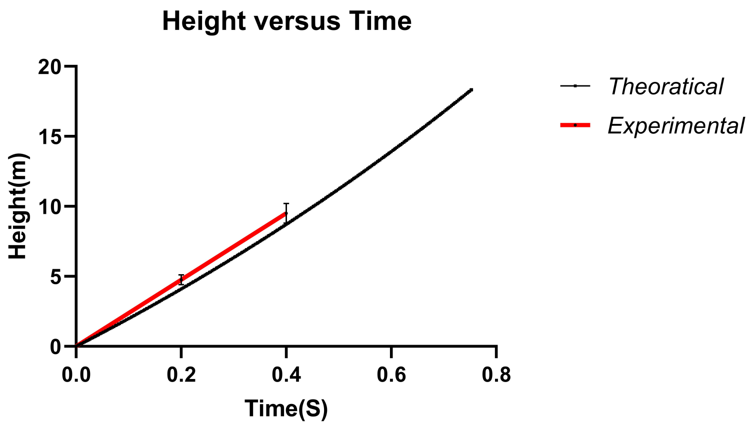

Figure E.2: Water filled up to 20%

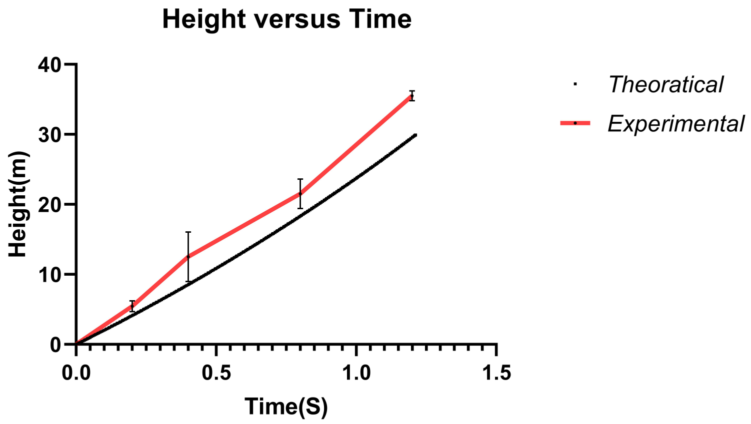

Figure E.3: Water filled up to 30%

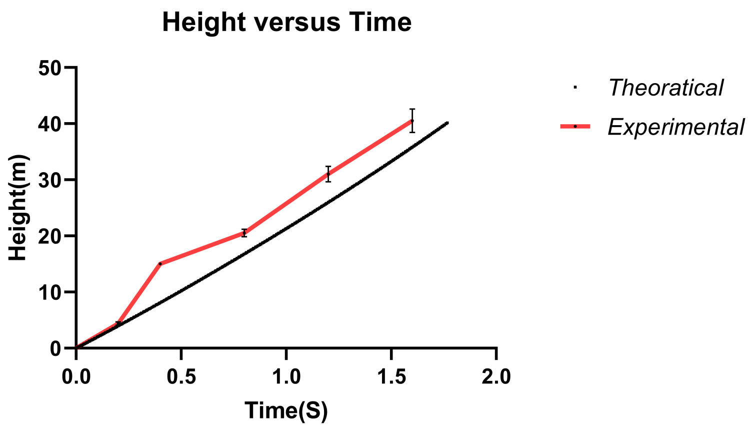

Figure E.4: Water filled up to 40%

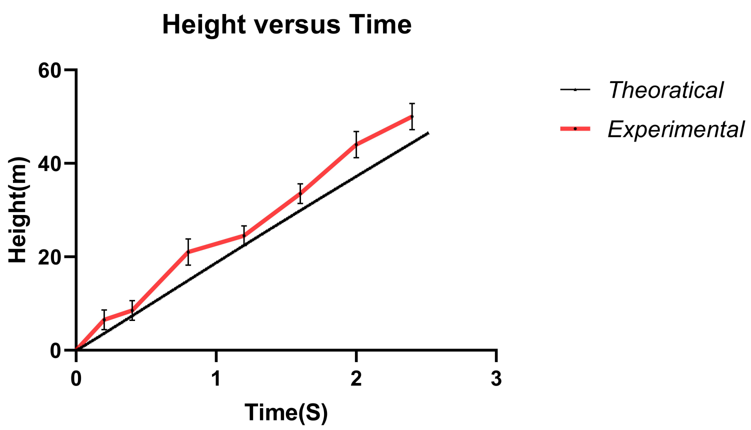

Figure E.5: Water filled up to 50%

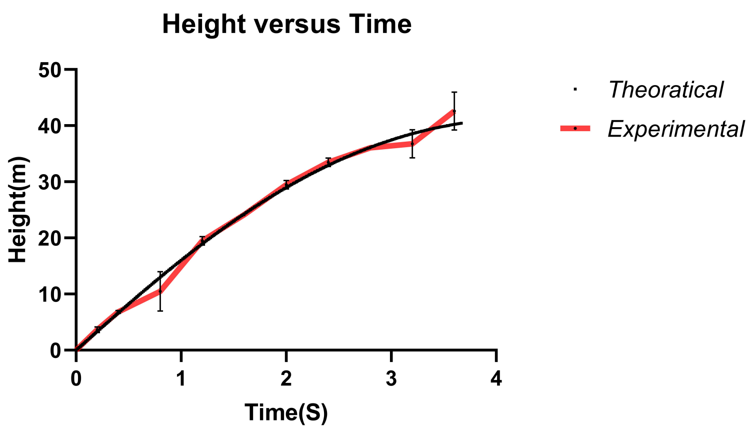

Figure E.6: Water filled up to 60%

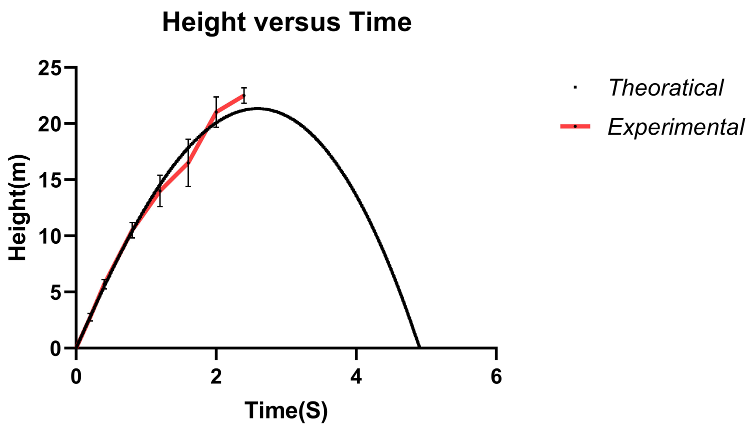

Figure E.7: Water filled up to 70%

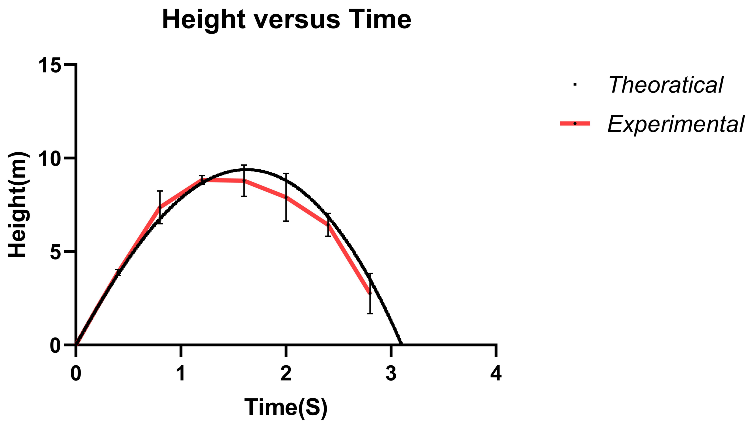

Figure E.8: Water filled up to 80%

Appendix F

Source Code

[1]

```
import java.awt.*;
import java.io.BufferedWriter;
import java.io.File;
import java.io.FileWriter;
import java.io.PrintWriter;
import java.util.*;
import jxl.Workbook;
import java.awt.Desktop;
import java.util.concurrent.TimeUnit;

import jxl.write.*;
import jxl.write.Label;

public class Main {

static double gamma = 1.4000;
static double densitywater = 1000.0000;
static double PressureAtmospheric = 101325.0000;
static double DeltaT = 0.0001;
static double Vtotal = 0.0020;
static double P0 =551580.0000;
static double dryMass = 0.4000;
static double Ae = Math.PI*(0.0025*0.0025);
static double K = 0.0003;
static double g = 9.8100;

static ArrayList<Double>outputHeight = new ArrayList();
public static double Pressure( double P_n, double Vair0) {
double Ve = Ae*Math.sqrt(2.0 * delataP)* DeltaT;
double deltaP = (P_n - PressureAtmospheric)/ densitywater) ;
```

```
double numerator = (gamma * P0 * Math.pow(Vair0, gamma)) * Ve;
\\
double epi = Math.pow(P0 * (Math.pow(Vair0, gamma) / P_n,(double)1;
double denominator =Math.pow((epi, (double)1 / gamma)), 1.0+gamma);

double result = P_n - (double)(numerator) / (double)denominator;
return result;
}
\\
public static double Mass( double P_n, double Vair0){
double exp1 = P0*Math.pow(Vair0, gamma)/P_n;
double Omega= Vtotal - Math.pow(exp1,(1/gamma)))+dryMass;
double mass  = densitywater*(Omega)
return  mass;
}
\\
public static double Velocity(double P_n, double v_n, double Vair0){
double alpha = (P_n-PressureAtmospheric)*Ae -K*v_n*v_n);
double Velocity = v_n+((2*alpha)/Mass(P_n, Vair0)-g)*DeltaT;
return Velocity;
}
\\
static double Height(double H_n, double P_n, double V_n, double Vair0){
double height = H_n+Velocity(P_n,V_n,Vair0)*DeltaT;
return height;
}
\\
public static double finalVolume(double Vair0){
double ratio = ((double)PressureAtmospheric/(double)P0;
double finalVolume = Vair0/(Math.pow(ratio),1.0/(double)gamma));
return  finalVolume;
}
\\
public static double finalPressure(double vair0){
double pressure = Math.pow((vair0/Vtotal),gamma)*P0;
return pressure;
}
\\
public static double InitialMass(double Vair0){
double Vwater = (Vtotal-Vair0);
double Mass = densitywater*Vwater+dryMass;
return  Mass;
}
\\
public static double WorkDone(double Vair0){
```

```
double states = P0*Vair0-finalPressure(Vair0)*Vtotal;
double pv = PressureAtmospheric*(Vtotal-Vair0);
double work = (double)1/(double)(gamma-1)*(states)-pv;
return  work;
}
\\
public static double InitialVelocity(double Vair0){
double fraction = 2*WorkDone(Vair0))/(InitialMass(Vair0));
double initialvelocity=Math.sqrt((double)(fraction);
return initialvelocity;
}
\\
public static void  Execute
( double P_n, double Vair0, double V_n, double H_n) throws Exception{
dpuble power = Math.pow(P0*Math.pow(Vair0, gamma)/P_n,(1/gamma));
double Condition = Vtotal - power;
double P_n_1;
double V_n_1;
double H_n_1;
double mass;
int i = 0;
FileWriter Output = new FileWriter(new File(localfile));
//localfile= insert your directory for output here
BufferedWriter OuputBuffer = new BufferedWriter(Output);
PrintWriter OutputFlush = new PrintWriter(OuputBuffer,true);
while(Condition > 0){
P_n_1 = Pressure(P_n, Vair0);
P_n = P_n_1;
mass = Mass(P_n,Vair0);
V_n_1 = Velocity(P_n,V_n,Vair0);
V_n = V_n_1;
H_n_1 = Height(H_n,P_n,V_n,Vair0);
H_n = H_n_1;
outputHeight.add(H_n);
OutputFlush.println(
"Pressure "+i+++" : "+P_n+"Height"+i+" :+H_n);
if(H_n >0){
 OutputFlush.println(H_n);
}
 Condition =  Vtotal - Math.pow(P0*Math.pow(Vair0, gamma)/P_n,(1/gamma)
}
}
}
\\
public static void main(String[] args) throws Exception {
double Vwater = 0.5*Vtotal;
```

```
double Vair0 = Vtotal - Vwater;
Execute(P0, Vair0, InitialVelocity(Vair0), 0);
File excel = new File("D:\\test.xls");
WritableWorkbook wworkbook;
try {
wworkbook = Workbook.createWorkbook(excel);
//Sheet name
WritableSheet wsheet = wworkbook.createSheet("First Sheet", 0);
//row 1
//Label label =new Label(0, 0, "Time");
//wsheet.addCell(label);
//label = new Label(1, 0, "Height");
//wsheet.addCell(label);
int row = 0;
int row2 = 0;
double time = 0.0001;
int k = 0;
for(  double height : outputHeight){
Label lab1 =new Label(0, row++, Double.toString(time));
wsheet.addCell(lab1);
time = time+0.0001;
Label lab2 =new Label(1, row2++, Double.toString((double)outputHeight.g
wsheet.addCell(lab2);
}
wworkbook.write();
wworkbook.close();
System.out.println("finished");
} catch (Exception e) {
System.out.println(e);
}
Desktop desktop = Desktop.getDesktop();
if(excel.exists()){
desktop.open(excel);
TimeUnit.SECONDS.sleep(15);
Runtime.getRuntime().exec("taskkill /IM excel.exe");
}\\ close application after 15 seconds
}
}
```

Bibliography

[1] Akash Amalan. Bsc computer science, Program : Java.

[2] Woodbank Communications. https://www.mpoweruk.com/rockets.htm, Date Accessed : 02-08-2019.

[3] Greg Finney. https://www.researchgate.net/publication/253753714. *Analysis of a water rocket, Honors Physics*, (10):1, Date Accessed : 03-08-2019.

[4] MIT. http://web.mit.edu/16.unified/www/FALL/systems/LabNotes/wrocket.pdf. *Water Rocket Calculations*, (10):1, Date Accessed : 15-08-2019.

[5] NASA. https://www.grc.nasa.gov/WWW/k-12/airplane/drageq.html. *The Drag Equation*, (10):1, Date Accessed : 15-08-2019.

[6] Rank. http://hyperphysics.phy-astr.gsu.edu/hbase/pber.html?fbclid=IwAR27E63-TGfk4IAlrKl9vLKRcbC2DXeayuqR9K2B-pTJ8QlFXXHOubyHUpE, Date Accessed : 02-08-2019.