Efficiency of the Slide Rule

Research question: How does the efficiency of the slide rule in vector calculations compare to that of the scientific calculator?

Extended Essay

Subject: Mathematics

Word count: approx. 3916

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August 2021

Contents

1. Introduction

Mathematicians and engineers may use different devices to complete calculations, which range in purpose and automaticity. One of these devices is the slide rule, a device consisting of a set of logarithmically graduated sliding scales, which when used in combination can perform a range of calculations, such as multiplication, division, square and cube roots (and their inverses), as well as basic trigonometric functions, using the sliding piece and cursor in combination with different scales. This simple device, my own slide rule pictured in figure 1, was used to send astronauts to the moon, and has surprising computational power when its format is compared with the advanced calculators used today (Stoll, 2006).

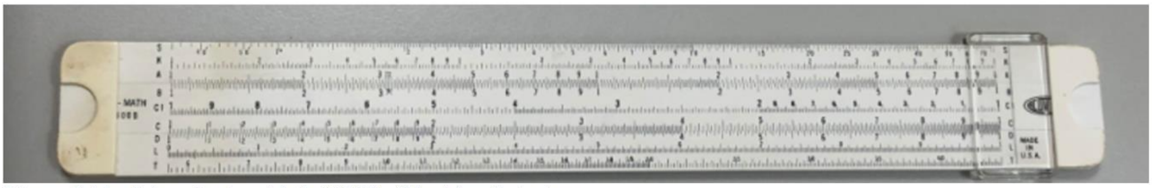

Figure 1 My slide rule, Acu-Math #400B of the Mannheim type

The slide rule is based on the logarithmic scale, to which John Napier is attributed its invention in 1614, while the first person to have used multiple logarithmic scales in combination to perform calculations is English mathematician Edmund Gunter (Britannica, 2019). The slide rule came into more widespread use in the latter half of the 19th century, after Amédée Mannheim's introduction of the slide rule with today's most common arrangement of scales (Britannica, 2019). There were few competitors for the slide rule among engineers as a handheld calculator until 1972, when the HP-35, the first handheld digital scientific calculator, was introduced; this device could perform "all the functions of the slide rule to ten-digit precision over a full two-hundred-decade range", performing logarithmic and trigonometric functions which were previously unique to the slide rule (ETHW, 2009). Consequently, over the course of the 1970s and with continued developments within affordable handheld calculators, the slide rule diminished significantly in popularity and is almost unused today (Store Norske Leksikon, 2006).

Considering the accomplishments made possible by use of this simple construction, I found myself asking the question of why the slide rule is not in use today, not by engineers nor students (Stoll, 2006).

To attempt to answer this question, I intend to investigate its efficiency, by comparing the slide rule's operation to the scientific calculator, such as the Casio fx-82ES PLUS pictured in figure 2.

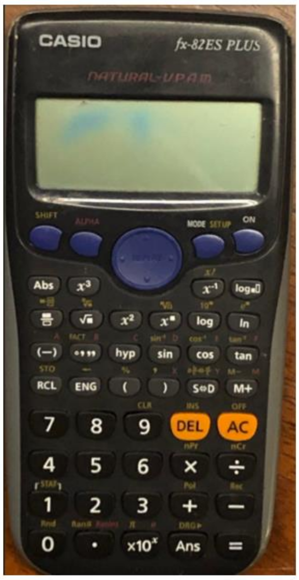

Figure 2 My Casio fx-82ES PLUS

Specifically, I will investigate the slide rule's efficiency in vector calculations because they are used in many areas of both science and mathematics, such as calculating forces in mechanics. Vectors require several different operations when being calculated and therefore give a comparison of multiple operations within a single general application. As a result of this research, an insight into the efficiency of the slide rule and possibilities for adjustments can be gained.

2. Operation of the slide rule

2.1 Basic manipulation

To be able to use the slide rule in vector calculations, a knowledge of its basic operations is necessary. The slide rule exists in many variants but is most often seen in versions similar to the one in figure 1. This slide rule uses 9 different scales, which, when used in combination, can perform a range of functions.

Perhaps used most frequently and with the most ease, are the C and D scales – in combination, these two can perform multiplication and division. The two scales are adjacent, where the C scale is located on the sliding part of the scale and the D scale is static on the lower part, as seen in figure 3.

For multiplication, the first factor of a two-factor calculation will be located on the D scale, and the C scale will be slid so that the index (left end of the rule, where the scale reads 1) aligns with the factor on the C scale (Acu-Math, n.d.). The second factor will then be located on the C scale, and the number on the D scale in accordance with this second factor will read the product (Acu-Math, n.d.). An example of this is seen in figure 3, when multiplying 4 by 2 to obtain 8 as an answer.

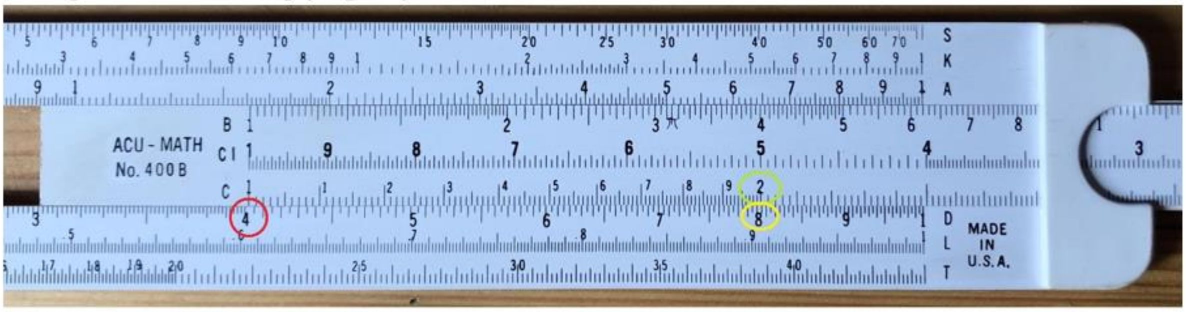

Figure 3: Multiplying 4 by 2 with a slide rule.

The calculations behind this operation are based on logarithms. C and D scales are identical scales incremented logarithmically (Pasquale, 2011). The distance from the index to any point on the scale can be regarded as a function of the number incremented: $f(x)$. This function can be defined as:

$$f(x) = \log(x) \tag{1}$$

For example, the number 3 would be placed at 0.447 of the physical length of the scale (figure 4), as

$$f(3) = \ log(3) = \ 0.447 \tag{2}$$

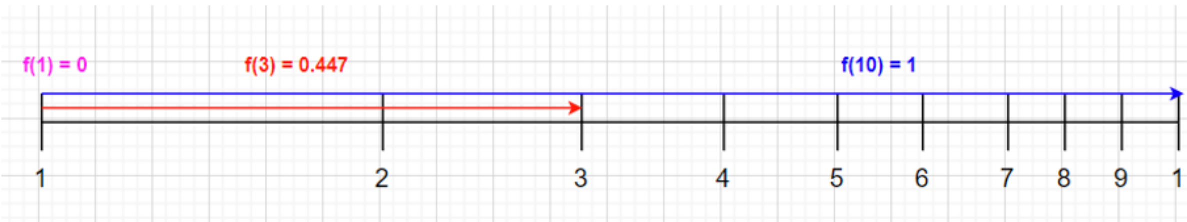

Figure 4: Example of a logarithmic scale

Applying this concept to the C and D scales used in combination, the functions $f(x)$ and $g(y)$ indicate the respective distances from the index for a value of x on the D scale or y on the C scale:

$$f(x) = \log(x) \tag{3}$$

$$g(y) = \log(y) \tag{4}$$

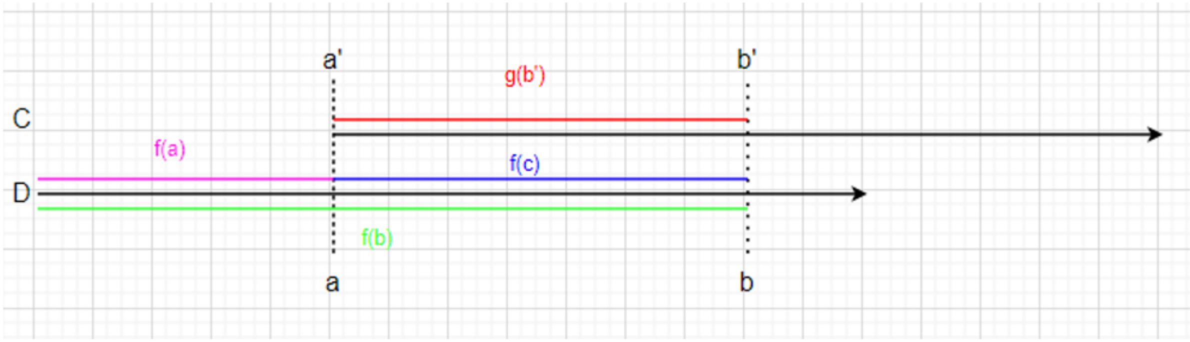

Figure 5 Intuition behind the slide rule

Figure 5 shows the adjacent C and D scales, where the C scale has been displaced by a factor of a on scale D. Point a on the D scale is in register with point a' on the C scale in a similar way to points b and b', but it is important to note that these are four distinct points due to the differing positions on their respective slides.

From the diagram in figure 5 it is clear that:

$$f(b) - f(a) = f(c) = g(b') \tag{5}$$

Hence,

$$f(a) + g(b') = f(b) \tag{6}$$

$$\log(a) + \log(b') = \log(b) \tag{7}$$

Applying the properties of logarithms:

$$\log(ab') = \log(b) \tag{8}$$

$$ab' = b \tag{9}$$

This corresponds to the operation done with the multiplication, where the slide was displaced by a factor of a, the second factor b' was identified, and the product was given by the resulting reading on the D scale, b.

The reverse operation can be used in a similar way with division, the inverse operation of multiplication. In the execution of a calculation for $\frac{b}{b'} = a$, the dividend (b) will be identified on the D scale and set in register with the divisor (b') on the C scale. The quotient, a, can be identified on the D scale in register with the index of the C scale.

One of the main limitations of the slide rule in even its most basic operations like multiplication, are the issues that arise in significant figures. The slide rule contains numbers on the scale from 1 to $9.\overline{9}$. As a result, numbers must be computed in their scientific notation: for example, 129 is computed in the same way that 1.29 would be – similarly, 0.00045 would be calculated in the same way as 4.5. When calculating vectors, this could lead to issues arising in accuracy, as this is an application of mathematics that often requires many steps rather than just one manipulation. Intermediate results in the calculation would need to be noted down manually and kept track of by the operator.

A handheld scientific calculator, such as the Casio fx-82ES PLUS, is able to store recent memory of answers acquired, while there is no such counterpart in any standard slide rule, leaving this aspect to be subject to human error. As a result, a reason for the preferred use of the scientific calculator over the slide rule is provided, and gives insight into the usefulness of the slide rule as a mathematical tool. This is especially noticed in multiple-step calculations, such as those done in calculating vectors, and will be further investigated after establishing a generalization of slide rule operation.

2.2 Generalizing the operation of the slide rule

A generalization of the operation of the slide rule is necessary to be able to apply it to vector calculations.

The central principle of the slide rule is the following equation, representing the difference of line segments as shown in figure 5 (Pasquale, 2011):

$$f(b) - f(a) = g(b') - g(a') \tag{12}$$

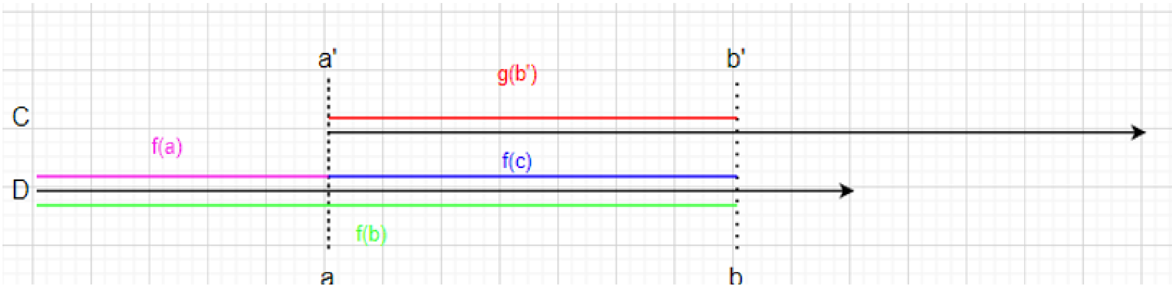

Figure 5 Intuition behind the slide rule

To isolate b as the solution for a given calculation:

$$f(b) = f(a) + g(b') - g(a') \tag{13}$$

$$b = f^{-1}\big(f(a) + g(b') - g(a')\big) \tag{14}$$

This can be expressed in more general terms by defining the result b as a function of x, y, z, such that $b = a(x, y, z)$,where the functions f and g are defined by the graduation of the respective scales (Pasquale, 2011).

$$a(x, y, z) = f^{-1}\big(f(x) + g(y) - g(z)\big) \tag{15}$$

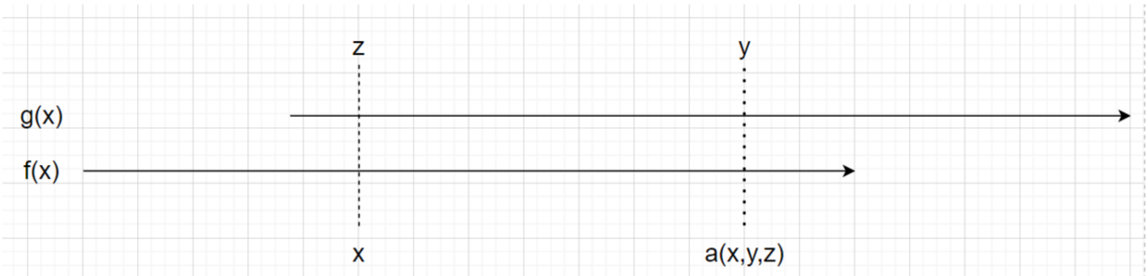

Figure 6: Visual representation of equation 14

With this generalization of the slide rule, its operation can be applied to calculations done in vectors.

3. Use of the standard slide rule in vector calculations

3.1 Determining the intersection of two three-dimensional lines

One problem commonly encountered in the calculation of vectors is determining the nature of intersection of three-dimensional lines. This will be attempted to be done using the slide rule and compared with the same calculation on a scientific calculator.

The vector form of the equation of a three-dimensional line may be represented in the following vector form (Roberts, 2007):

$$(x, y, z) = (a, b, c) + m(d, e, f) \tag{16}$$

where m is a variable constant (the parameter), (a, b, c) represents the position vector and (d, e, f) represents the directional vector (Roberts, 2007).

Solving for the case of when these two lines are parallel is quite straightforward in the use of the slide rule. Three-dimensional vector lines are parallel if the directional vectors are proportional, i.e., multiplied by some constant k (Roberts, 2007):

$$(d_1, e_1, f_1) = k(d_2, e_2, f_2) \tag{17}$$

When using the slide rule to calculate this, proportions can easily be represented using the slide rule using the C and D scales.

The C and D scales are identical, logarithmically graduated scales. They will be defined as f(x) and g(y), such that:

$$f(x) = \log(x) \tag{1}$$

$$g(y) = \log(y) \tag{2}$$

Applying equation (15) (Pasquale, 2011):

$$a(x, y, z) = f^{-1}\big(f(x) + g(y) - g(z)\big) \tag{15}$$

$$a(x, y, z) = 10^{\log(x)+\log(y)-\log(z)} \tag{18}$$

Applying properties of logarithms:

$$a(x,y,z) = 10^{\log\left(\frac{xy}{z}\right)} \tag{19}$$

$$a(x,y,z) = \frac{xy}{z} \tag{20}$$

$$\frac{a(x,y,z)}{y} = \frac{x}{z} \tag{21}$$

With this same concept, the components of each directional vector in r_1 and r_2 can be evaluated in a similar way. If the two lines are parallel, the following relationship is present:

$$\frac{d_1}{d_2} = \frac{e_1}{e_2} = \frac{f_1}{f_2} \tag{22}$$

Therefore, if d_1 is identified on the C scale and the D scale is displaced such that it is in register with the d_2 on D, e_1 and e_2 and f_1 and f_2 should therefore also be in register with each other, respectively, as the same proportional relationship is present. This case is illustrated in figure 7. If these do not align, it can easily be identified that the lines are not parallel. It is evident that using the slide rule in this instance is very efficient, and can be done with one slide, whereas each proportion would need to be inputted separately to the scientific calculator.

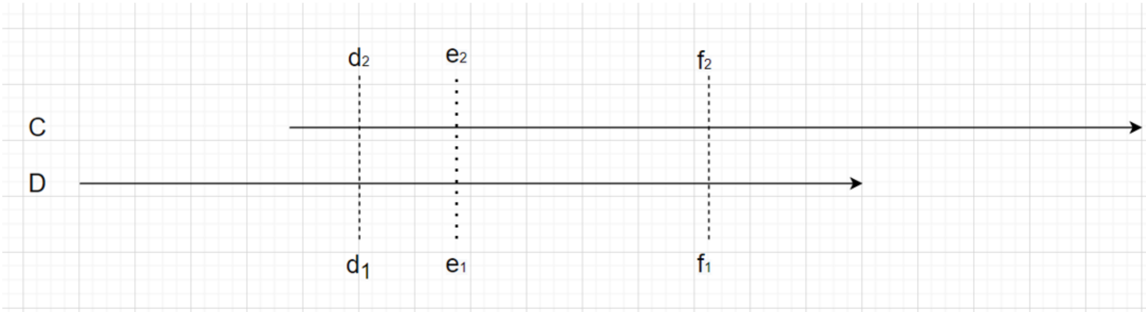

Figure 7: Components of vectors of two parallel lines in 3D space

If lines in 3D space are not parallel, they either intersect or are skew. There are several different ways to approach solving the situation of lines not being parallel, although most involve the use of matrices after converting to parameter form. In this process, simultaneous equations are used to determine a point of intersection or to clarify their skewness (Roberts, 2007)

Reproducing a method like this is quite difficult to do with the use of the slide rule, as the solution of simultaneous equations involves addition and subtraction, while the slide rule is not well suited for these operations without significant adjustments, such as those which will be expanded on in section 6. Approaches which do not involve the use of simultaneous equations include approaching it in its vector form: however, a similar problem would be encountered when completing the cross product of vectors and computing matrices – addition and subtraction is involved here and could be done much easier using a scientific calculator where addition and multiplication are easily accessible. This provides an instance in which the scientific calculator's efficiency exceeds that of the slide rule.

3.2 Converting to polar coordinate form

Although the slide rule's efficiency lacks in comparison to that of the scientific calculator in operations involving matrices and extensive addition and subtraction, there are other operations where the slide rule may be of value in the calculation of vectors.

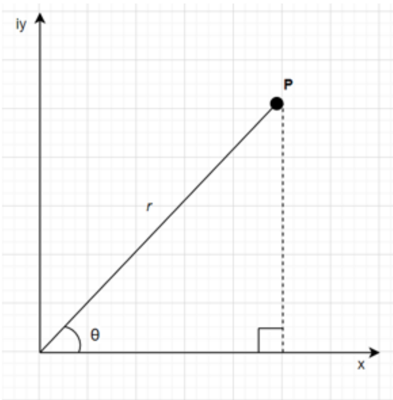

Figure 8: Polar form of a complex vector

A common operation done with vectors is the conversion to polar coordinates, often used in the complex plane and in combination with Euler's formula, for example in the solution of differential equations. Converting a complex number from its Cartesian form to its polar coordinate form can be represented by the following equation (Roberts, 2007):

$$x + iy = r(\cos\theta + i\sin\theta) \tag{27}$$

where this complex number in the form $x + iy$ is represented by vector at a point P ($\overrightarrow{OP}$) with coordinates (x, y) and polar coordinates (r, θ), where r is the magnitude of the vector and θ is the angle it makes with the x-axis, as seen in figure 8.

The length of the vector, r, is found using Pythagoras' theorem:

$$r = \sqrt{x^2 + y^2} \tag{28}$$

This can be found by using the A scale in combination with the C or D scales, used to find squares and square roots. This calculation would require the addition of x^2 and y^2 on paper, which is not well-suited for the slide rule; this will be investigated further in section 5.

Angle θ can be defined in terms of x and y:

$$\tan\theta = \frac{y}{x} \tag{29}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{30}$$

When calculating the tangent and inverse tangent functions on a slide rule, there are two cases which must be considered. The D and T scales are used in combination to find the tangent of an angle, or to find the angle given its tangent. However, the T scale only encompasses values up to 45°, and therefore is operated differently for two distinct cases.

For cases when $x \geq y$:

$$\frac{y}{x} \leq 1 \tag{31}$$

$$tan\theta \leq 1 \tag{32}$$

$$\theta \leq 45° \tag{33}$$

In this case, the angle can easily be found on the T scale, after using a combination of C and D scales. As in standard division, y will be divided by x by identifying y on the D scale, aligning this in register with x on the C scale and identifying the result on D in register with the index of the C scale, as explained in section 2.1. This point on D will then read the value for $\tan\theta$. The angle may be identified on scale T in register with the previous reading on D.

In the outstanding case, when $x < y$, it follows that $\theta > 45°$ and the following identity must be used:

$$\cot\theta = \tan(90° - \theta) \tag{34}$$

Here, $\cot\theta$ can be found by simply reversing the division, such that the C and D scales are used to find the reciprocal. The same process will then be done as described in the previous case to divide x by y (rather than y by x), which will yield a value for $\cot\theta$ on D. The value of θ will be identified on T, which will then be subtracted from 90 to find the final angle for the principal angle between 0 and 90°, according to the identity in equation (34). This angle will then need to be adjusted according to its location in the four quadrants of the complex plane, adding another step to this process.

The slide rule can be used in this example of vector calculations of finding the polar form of a point in the complex plane. However, it has several limitations. In finding r, not all calculations can be done entirely on the slide rule, with some steps requiring the use of pen and paper. This is not the case for the scientific calculator, where every step in the calculation can be done using the device.

The value for θ would need to be manually adjusted to correspond to the location of the point in the complex plane, as negative numbers may not be computed using the slide rule; only acute angles can result. Also, as previously discussed, when using the slide rule in multiple step-calculations, significant figures and numbers must be kept track of manually. This adds possibility for errors if the results obtained from the above calculations were to be further manipulated on the slide rule, which is not uncommon in the application of vectors.

3.3 Calculating the dot product of two vectors

To further compare the efficiency of the slide rule with that of the scientific calculator, the slide rule can be attempted to be applied to other calculations done with vectors.

A common calculation done is taking the scalar product, or dot product, of two vectors. Vectors a and b of dimension n are defined as:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \tag{35}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} \quad (36)$$

The dot product of these two can be found algebraically as follows (Roberts, 2007):

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \quad (37)$$

This clearly involves the addition of many terms. Calculating this using a scientific calculator may be time-consuming but can be done easily as many terms may be added in a single calculation. However, there is no corresponding method for the slide rule, as the device cannot calculate many terms at once, and can input up to only three variables, (x, y, z), for each manipulation, as seen in equation (15) (Pasquale, 2011):

$$a(x, y, z) = f^{-1}\big(f(x) + g(y) - g(z)\big) \quad (15)$$

Addition and subtraction could be performed on the slide rule by the key concept of addition of line segments, if the graduation of incrementation of the two scales used were directly proportional to the distance, such that:

$$f(x) = x \quad (38)$$

$$g(y) = y \quad (39)$$

$$a(x, y, z) = x + y - z \quad (40)$$

However, most slide rules do not have more than one linear scale defined in this manner – the Mannheim slide rule, for example, has one linear scale, L, which is used in combination with the logarithmic scales to operate conversions to logarithms (Acu-Math, n.d.).

As a result, proportions must be used to complete addition and subtraction, for example by using the following concept in the addition of x and y (Nikitin, n.d.):

$$\log(x + y) = \log\left(y\left(\frac{x}{y} + 1\right)\right) \quad (41)$$

$$= \log(y) + \log\left(\frac{x}{y} + 1\right) \tag{42}$$

Even so, this approach would require the separate addition of 1, which adds further steps and areas for error.

In this instance of calculating the dot product of vectors, the shortcomings of the slide rule are clear. This provides mathematical intuition about the functions of the slide rule: for calculations with multiple steps and many inputs, other devices such as the scientific calculator or even doing the calculations by hand, are more suited. The calculations are not necessarily more complex, which may suggest that the use of the slide rule is dependent on the magnitude of calculations, rather than the complexity, which is clear from other manipulations. For example, finding the angle in polar coordinates is not something that could be done easily mentally, but is done with ease using the slide rule. As a result, with this mathematical knowledge of the operation of the slide rule, developments to the slide rule can be investigated.

4. Developments to the slide rule for more efficient use in vectors

As previously noted, vector calculations can be done using existing scales, but include several calculations that cannot be done easier than the scientific calculator – for example, calculating the dot product algebraically or finding the length of a vector. To research this further and evaluate the efficiency of the slide rule specifically with vectors, it may be of use to introduce possible new scales which follow the methods of the slide rule.

An example of a combination of scales that could calculate common problems encountered with vectors, is finding the length of a two-dimensional vector, $\begin{bmatrix} x \\ y \end{bmatrix}$, using Pythagoras's theorem, in a similar way to what was seen in the calculation of polar coordinates. If functions f and g, representing the graduation of two new scales, are defined such that:

$$f(x) = x^2 \tag{43}$$

$$f^{-1}(x) = \sqrt{x} \tag{44}$$

$$g(y) = y^2 \tag{45}$$

The function of this operation would be defined as follows, according to the definition in equation (15) (Pasquale, 2011):

$$a(x, y, z) = f^{-1}\big(f(x) + g(y) - g(z)\big) \tag{46}$$

$$a(x, y, z) = \sqrt{x^2 + y^2 - z^2} \tag{47}$$

In this case, variable z is not needed for the calculation due to the absence of a third variable in the calculation. Point z will therefore be defined as the index of f, such that $f(z) = 0$ and therefore $z = 0$ in this case (Pasquale, 2011).

$$a(x, y, 0) = \sqrt{x^2 + y^2} \tag{48}$$

An example of this can be seen in figure 9, where the following calculation is done by using these new scales, and the variables (x, y, z) are defined in the manner as seen in equation (15).

$$a(3,4,0) = \sqrt{3^2 + 4^2} = 5 \tag{49}$$

The first factor $(x = 3)$ is located on the lower slide and the index of the upper scale $(z = 0)$ is placed in register with this. The result $(a(3,4,0))$ is located on the lower scale in register with the second factor $(y = 4)$ on the upper scale, as seen in figure 9.

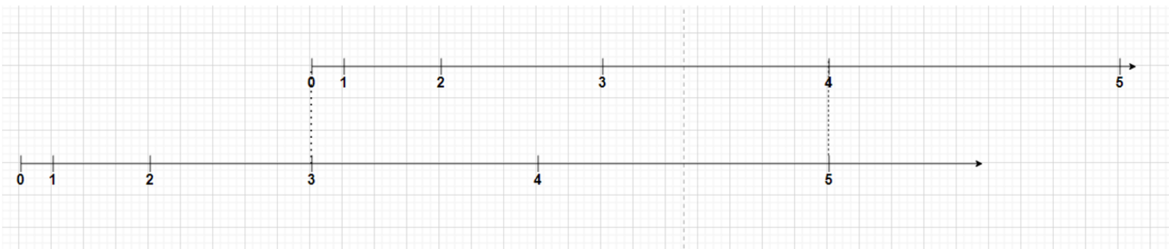

Figure 9: New, quadratically graduated slide rule scales

Here, the calculation can be done with one slide of the slide rule by using these two new scales in one single slide, efficiently yielding an answer. The issue here arises in that the standard slide rule has a limited set of scales at its disposal. Although these calculations could be done much easier with a hypothetical slide rule with the scales defined above in equations (43) and (45), the common slide rule does not bestow these scales and would therefore need to be constructed.

When this adjustment has been made, however, it enables easier computation of other vector-related calculations. For example, calculating the dot product of vectors can be done geometrically, as opposed to the algebraic method outlined in section 5. For two vectors a a and b of dimension n:

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{bmatrix} \tag{34}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix} \tag{35}$$

The dot product (scalar product) of these two vectors can be found geometrically, using the slide rule. It is defined as follows (Roberts, 2007):

$$a \cdot b = \|a\| \|b\| \cos \theta \tag{50}$$

Where $\|a\|$ and $\|b\|$ each denote the magnitude or length of the respective vectors, and θ denotes the angle between the two. The magnitude of each vector can be found using the new scales previously explained, and from there be multiplied with each other and the cosine of θ which is easily computed using already existing scales, to find the dot product.

Even within the same calculation of the dot product, this demonstrates how the efficiency of the slide rule varies greatly compared to the scientific calculator, depending on whether the geometric or algebraic approach is taken – the scientific calculator is of far easier use when using the algebraic approach and calculating the sum of many terms, while its efficiency does not far exceed that of the slide rule when using the geometric approach.

This indicates that the slide rule is not suited from calculations with many terms, or of bigger computational magnitude. While the scientific calculator is better suited for inputting multiple numbers in a clear way, this overview is not available using the slide rule, where it is evident that only three inputs can be used at a time. Therefore, when approaching this same vector calculation of dot products from a different angle which uses more direct steps and fewer terms, the slide rule is indeed more efficient, and could be more efficient than its modern counterpart, the scientific calculator. However, this was only possible after adjustments were made to the scales available on the slide rule, but nonetheless demonstrate the central principles of slide rule operation and its efficiency.

5. Conclusion

The slide rule is an example of the way in which different calculators uses different mathematical approaches to compute the same calculations. In comparison with the scientific calculator, there are differences in not only the efficiency of the computation but in the foundational manner in which the computations are approached. There are some calculations in which the slide rule may be more efficient, such as in the operation of certain calculations relating to vectors, like identifying parallel lines. However, the main flaw of the slide rule appears to be that its nature and efficiency can be generalized to certain types of operations, such as proportions, but not to broader areas of mathematics, such as vectors.

Furthermore, many types of calculations beyond standard multiplication and division require the construction of new scales on the slide rule, if to be done efficiently, as seen in calculating the magnitude of a vector. This inconsistency may provide a reason for its lack of use today, when compared with the scientific calculator which is more generally efficient in a range of situations and uses. Even within the single topic of vectors in mathematics, a knowledge of its operation in different calculations and an understanding of its nature in applying it to solve its problems in new ways, is required. This may be because the input to a scientific calculator matches very well to the written calculation, while inputting these calculations to the slide rule holds a vastly different format and therefore requires a thorough knowledge of the device’s operation.

Nonetheless, this application of the slide rule in vector calculations provides an insight into how mathematics, such as the laws of logarithms, can be used to solve calculations from a different approach. Perhaps the slide rule is useful not for its universal efficiency or accessibility as a general mathematical tool in, for example, vector calculations, but for its value as a way of understanding mathematics from a different angle than the vastly different scientific calculator.

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