# The perfect squat

*Julia Lise Boccadoro*

*19 pages*

# Contents

# Introduction

I have been going to the gym to weight train several times a week for the last four years, and through my interest in weight training, I have noticed that a surprisingly large amount of people, including me, struggle with doing a weighted squat with proper form. I have on occasion watched as gym plates have been used to elevate the heels of the person squatting, making the person's form better. This observation sparked curiosity in me about the role of body proportions and joint flexibility in doing a squat properly, and under what conditions a block is needed, thus, my research question; **How do body proportions influence the ability to do a perfect squat?**

One of the most widely practiced weightlifting exercises is the weighted squat; "sitting down on an invisible sturdy elevated surface, like a chair, and then standing up, but with weights" (TAYLOR, 2020). It is a full-body exercise that trains the gluteus maximus, the core muscles and quadriceps, along with calf muscles. In this paper, I will investigate how specific body length ratios (the lower leg, upper leg and back) influence the angles required at the ankle and hip to perform a correct squat. Then, I will determine whether it is necessary to aid the squat using a block to raise the heel off the floor, and if so, what the block height should be. All of the above will be made easy for any person to figure out with their own body proportions through a computer program I plan to make using python. The software aims to compute the necessary angle of the back during a squat and, if necessary, the height of the heel block used to aid the squat, from data input of particular body lengths and flexibility restrictions.

This investigation will provide insights into how individual body ratios influence movement in a squat and the potential adjustment required to optimize one's form to avoid injury and maximize effectiveness.

# Part 1; Who can do a perfect squat?

## Aim and clarification of task:

This part of the investigation will dive into how three body segment lengths (the lower leg, the upper leg and the back) along with the flexibility of the ankle, affect the upper body forward lean during a squat. This is important because it is necessary to restrict forward lean to avoid injury, making an unaided correct squat unfeasible for some people depending on their body proportions and flexibility.

A correct squat in this investigation is defined by three constraints when at the lowest point of the squat:

1. The weight/bar must be directly above the heels.
2. The upper leg must be parallel to the ground, as the training effect of the squat would not be as effective for muscle development if one stopped before that point.
3. To perform the squat while avoiding injury, the back angle relative to the upper leg must be above 60°. This is to prevent too much forward lean; executing the exercise with more lean than 60° would result in the small of the back being bent, which can lead to injury.

These restrictions can be observed in figure 1 below.

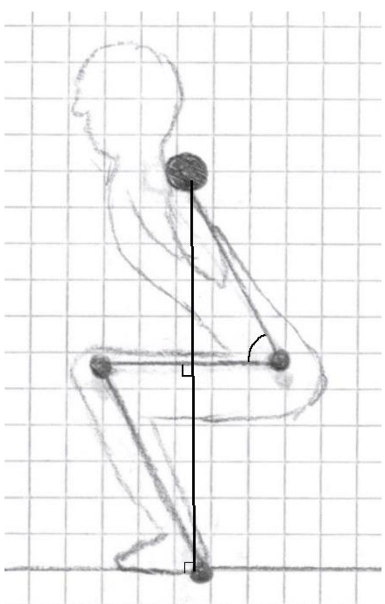


*Figure 1: Squat restrictions.*

Several key body measurements must be clarified and defined: the lower leg (*a*) is a measure in meters from the heel to the knee joint, the upper leg (*b*) is a measure in meters from the knee joint to the hip joint, and the back (*c*) is a measure in meters from the hip joint to the shoulders where the weight lies. The smallest ankle angle obtainable (*α*) is the angle measured in degrees (using a protractor) between the lower leg (*a*) and an imagined line between the heel and the ball of the foot (d). The back angle relative to the upper leg (β) is the angle in degrees between the back (*c*) and the upper leg (*b*) (also measured using a protractor). The key measurements *a, b, c, α* and *β* are visually shown in figure 2 below. The way in which measurement *d* is made is shown in figure 3.

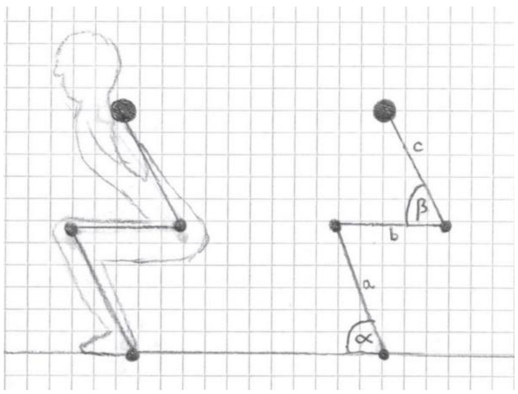



*Figure 2: A visual representation of variables used.*

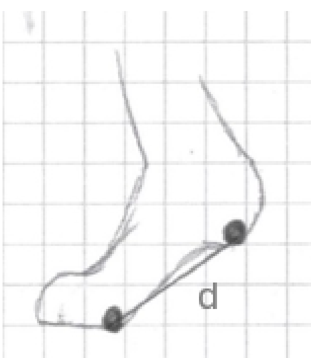



*Figure 3: Obtaining measurement d.*

## Developing the formula

In this section, I will develop an equation which will determine the back angle relative to the upper leg (*β*) required for a weighted squat, given specific body measurements (*a, b, c*) and the minimum ankle angle (*α*) a person can achieve. This will clarify the relationship between the variables along with specifying if those particular body measurements make it possible to perform a squat properly or not.

Figure 2 above in which the body is shown as segments, can also be imagined on a graph, in which the heel and the weight/top of shoulders are both at *x=0*, and where the heel is called point *A*, the knee joint point *B*, the hip joint point *C* and the weight/top of shoulders, point *D*:

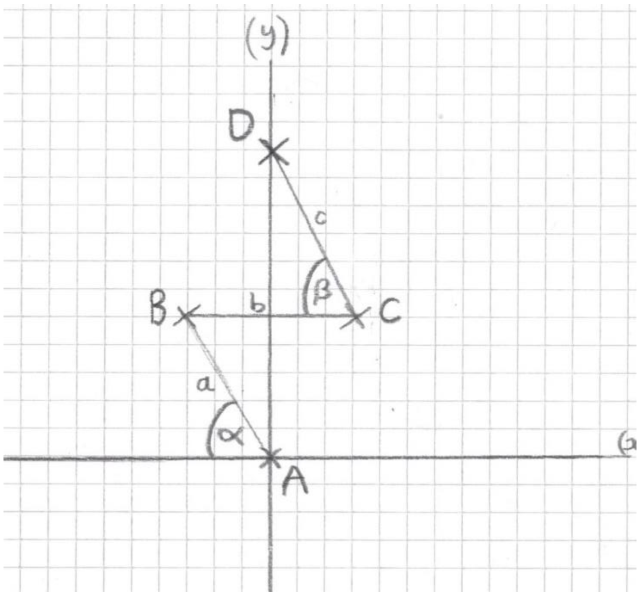
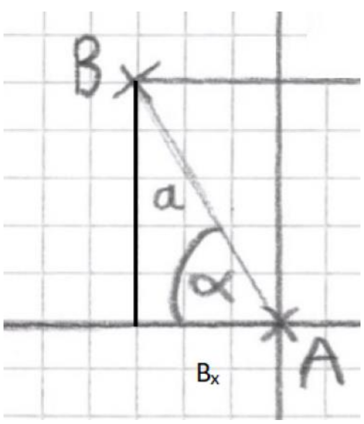



*Figure 4: Trigonometric construction from heel position (A) to shoulder (D).*




$$\cos\alpha = \frac{B_x}{a}$$

*Figure 5: example showing trigonometric construction to calculate $B_x$.*

From figure 4, given the x-coordinate 0 of the heel (*A*), the position of the knee (*B*) relative to the heel can be determined. The trigonometric construction is illustrated in figure 5. The horizontal displacement of the knee (*B*) will influence the position of the hip joint (*C*) and, subsequently, the back angle (*β*). From the coordinate system, step by step, the x-coordinates of each point can be found building on the last with help of right-angle trigonometry.

$$A_x = 0$$
$$B_x = a \cos \alpha$$
$$C_x = a \cos \alpha - b$$
$$D_x = a \cos \alpha - b + c \cos \beta$$

Since $D_x$ is also equal to 0, the equation of point $D_x$ can be written as:

$$0 = a \cos \alpha - b + c \cos \beta$$

Angle $\beta$ is responsible for determining whether or not the squat can be completed, as it is the variable on which a restriction is imposed for the squat to be done with good form, and the body lengths/the smallest ankle angle obtainable are people's unchangeable variables.
Therefore, the equation is rearranged for $\beta$:

$$0 = a \cos \alpha - b + c \cos \beta$$
$$-a \cos \alpha + b = c \cos \beta$$
$$\frac{-a \cos \alpha + b}{c} = \cos \beta$$
$$\boldsymbol{\beta = \cos^{-1}\left(\frac{-a \cos \alpha + b}{c}\right)}$$

This last equation is the equation needed to determine the hip angle required for the squat, given specific body measurements *a, b* and *c*, and the minimum ankle angle ($\alpha$) a person can achieve. Note that this equation is however only true for $\beta$>60°, as that is one of our restrictions. If $\beta$<60°, the squat is therefore not feasible.

## Program/code

To facilitate the application of the formula to determine the hip angle required for a squat, I made a program on python, in which the user can type in the lengths of their lower leg, upper leg, and back and the smallest ankle angle they can obtain. The program will then apply the equation above and return the back angle required.

This is the code:

```
from math import acos, cos, sin, pi

def DegreesToRadians(deg):
    return deg*pi/180

def RadiansToDegrees(rad):
    return rad/pi*180

a= float(input("Enter the length of your lower leg, in meters:"))
b= float(input("Enter the length of your upper leg, in meters:"))
c= float(input("Enter the length of your back, in meters:"))
Alphadeg= float(input("Enter the smallest ankle angle you can obtain, in degrees:"))

Alpha= DegreesToRadians(Alphadeg)

Beta= acos((-a*cos(Alpha)+b)/c)
Betadeg= RadiansToDegrees(Beta)
print("The angle your back should be relative to your upper leg is:", Betadeg)
```

This is a demonstration of the program's "receiver end", in which the standard example lengths and ankle flexibility are used and inputted, and the back angle $\beta$ is successfully returned:

```
Enter the length of your lower leg, in meters:0.45
Enter the length of your upper leg, in meters:0.45
Enter the length of your back, in meters:0.50
Enter the smallest ankle angle you can obtain, in degrees:50
The angle your back should be relative to your upper leg is: 71.246872614147
```

## The effect of each body length on the feasibility of the squat

This section aims to investigate the effect each variable has on the $\beta$ angle, as a beta angle smaller than 60°, as mentioned earlier, leads to the squat not being feasible. A graph will be plotted for each variable in terms of $\beta$ to see what numbers for *a, b* and *c* make the squat possible.

Note that the non-changing variables are fixed at the standard previously used, where *a*=0.45m, *b*=0.45m, *c*=0.50m and *α*=50°. These measurements come from a male adult person who volunteered to supply his measures for this paper.

## Lower leg length, *a*

This is the graph demonstrating the relationship between the lower leg length (*a*) and the back angle (*β*):

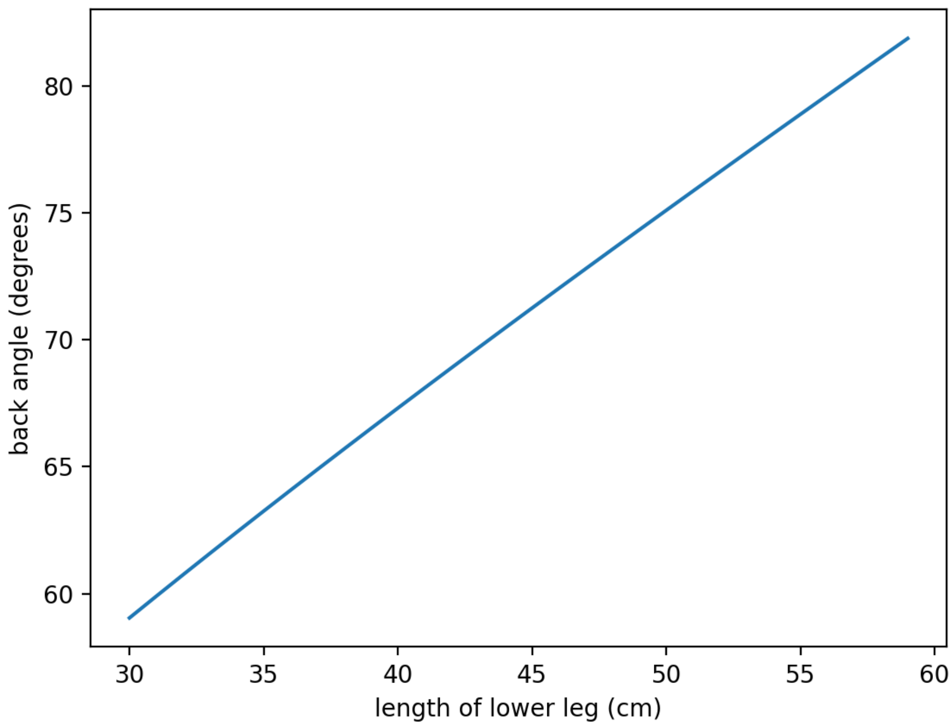



*Figure 6: relation between lower leg length and back angle.*

This increasing graph in figure 6 shows that the lower leg length and the hip angle seem to be proportional, and that the hip angle increases as the lower leg length increases. A long lower leg would increase the chances of being able to naturally perform a squat. It can be seen that the point at which the angle of the hip is less than 60 is when the lower leg is below 50cm, given the other standard measurements.

Back angle has to be 60 degrees or above, and was thus chosen as the minimum. The independent variable range was made to correspond with the back angle from 60 degrees and above.

## Upper leg length, *b*

This is the graph demonstrating the relationship between the upper leg length (*b*) and the hip angle (*β*):

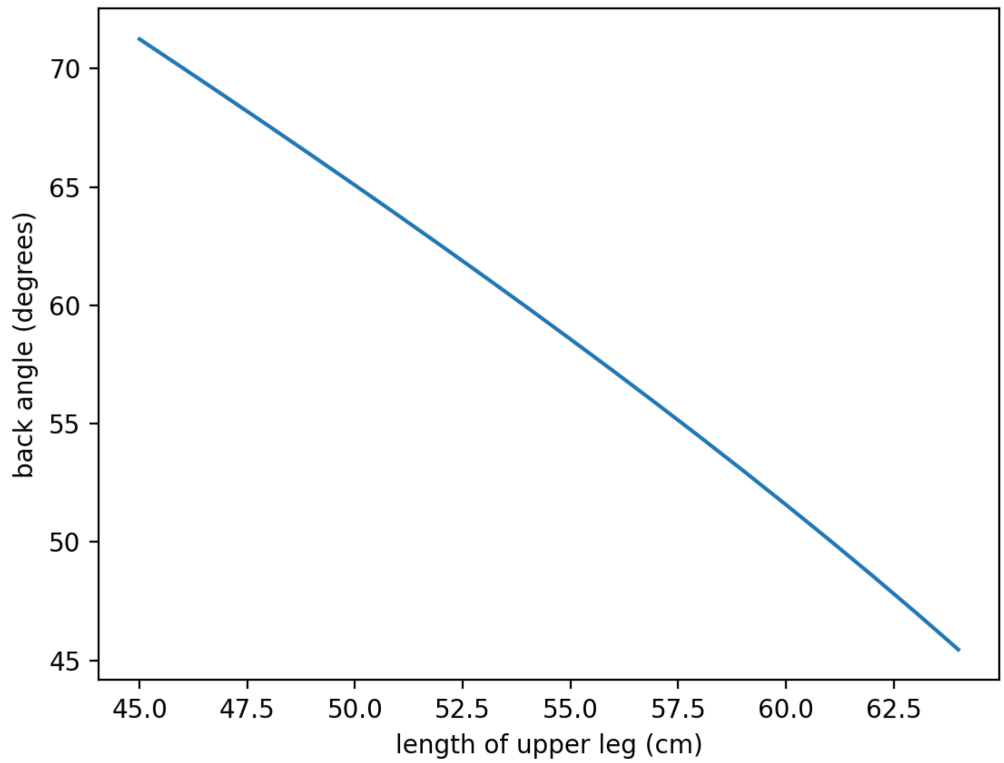



*Figure 7: relation between upper leg and back angle.*

This graph in figure 7 is decreasing and shows that the hip angle decreases as the lower leg length increases. A short upper leg would increase the chances of being able to naturally perform a squat. It can be seen that the point at which the angle of the hip is less than 60 is when the lower leg is below 46cm, given the other standard measurements.

Back angle has to be 60 degrees or above, thus the independent variable range was made to correspond with the back angle from 60 degrees and below.

## Back length, *c*

This is the graph demonstrating the relationship between the back length ($c$) and the hip angle ($\beta$):

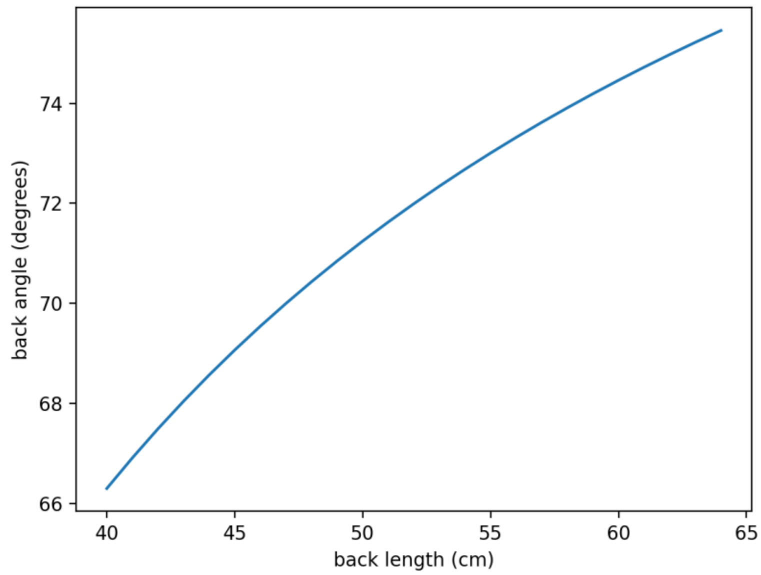



*Figure 8: relation between back length and back angle.*

This graph in figure 8 is an increasing function, and shows that the hip angle increases as the lower leg length increases. A long back would increase the chances of being able to naturally perform a squat. It can be seen that the point at which the angle of the hip is less than 60 is when the back length is shorter than 60cm, given the other standard measurements.

Back angle has to be 60 degrees or above, thus, the independent variable range was made to correspond with that back angle.

# Part 2; Extension - help of a heel block

## Aim and clarification of task

Using the equation above, I could evaluate whether the back angle ($\beta$) condition of 60° is met and, if not, calculate the block height required to compensate for insufficient ankle flexibility.

An equation for the block height needed can be found by adjusting the effective ankle angle to ensure the back angle constraint is satisfied. The block height will depend on the distance between the heel and the ball of the foot (foot's contact point on the ground), a segment previously defined as "*d*". This is therefore a new variable that must be provided by the individual. Below is a visual reminder of the what the measure "*d*" corresponds to:

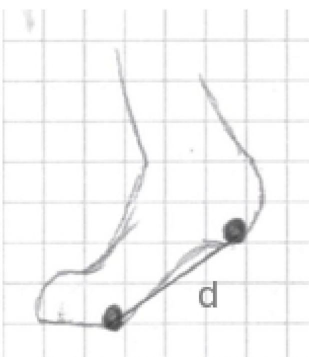



*Figure 9: Measuring the distance between the heel and the ball of the foot.*

While α becomes the full angle of the foot, there are two new variables that come into play in this section. These are $\theta$, which is the smallest ankle angle achievable (and was α when the foot was flat on the ground), and $\gamma$, which is the angle added to the initial ankle angle by the block. *h* is the height of the block. A full graph of the new setup and variables is shown below:

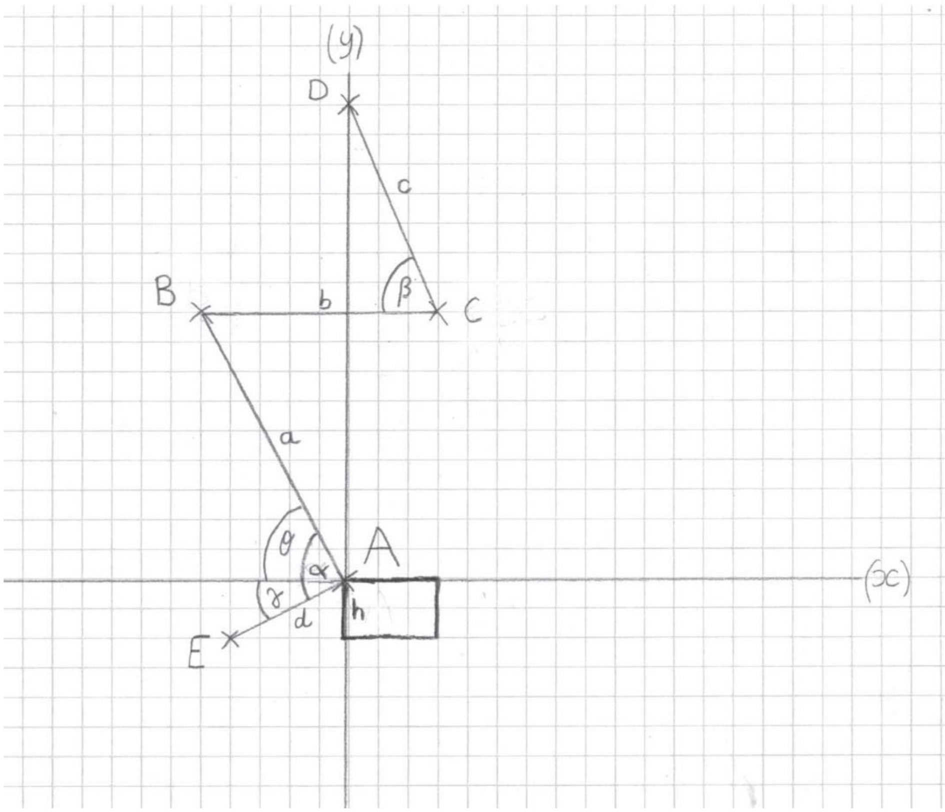



*Figure 10: Modified figure 3 to account for effect of heel block.*

The diagram is not quite realistic due to the fact that the heel would be floating in the air, only touching the corner of the block. Conveniently, the impact of the heel having to rest on the block further back then on the graph makes a very small negligeable difference to the results, due to the way the foot is built. This is demonstrated below:

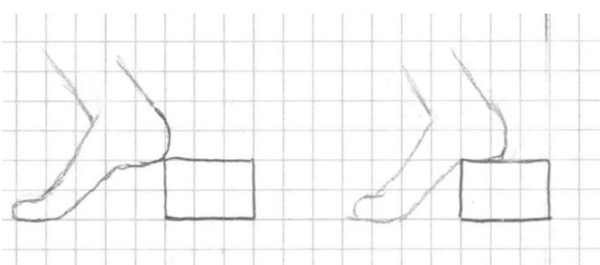


*Figure 11: details of heel block.*

It can be seen that the slope of the foot does not change when the heel rests on the block rather than when it touches its corner. This is due to the underneath of the foot not being a flat segment.

## Developing the formula

Bringing back the non-rearranged formula for $\beta$ made in Part 1, but replacing $\alpha$ by $\theta$ due to redefining of the variables,

$$0 = a\cos\theta - b + c\cos\beta$$

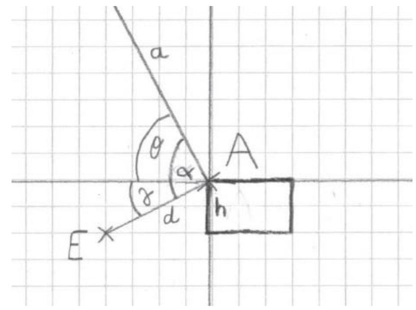



*Figure 12: heel block trigonometric construction.*

And the fact that;

$$\alpha = \gamma + \theta$$

$$\theta = \alpha - \gamma$$

$\theta$ can be replaced in the $\beta$ equation by $\alpha - \gamma$:

$$0 = \mathrm{a}\cos(\alpha - \gamma) \ -b + c\cos\beta$$

This can be rearranged for $\gamma$;

$$b - c\cos\beta = \mathrm{a}\cos(\alpha - \gamma)$$

$$\frac{b - c\cos\beta}{a} = \cos(\alpha - \gamma)$$

$$\cos^{-1}\left(\frac{b - c\cos\beta}{a}\right) = \alpha - \gamma$$

$$\alpha - \cos^{-1}\left(\frac{b - c\cos\beta}{a}\right) = \gamma$$

Given angle $\gamma$, *h* can be found by using the segment of the foot (*d*), and applying trigonometry:

$$\sin(\gamma) = \frac{h}{d}$$

$$h = d \times \sin(\gamma)$$

Given that $\beta$'s minimum is 60°, the full equation for the height (*h*) of the block would then be;

$$\mathbf{h = d \times \sin\left(\alpha - \cos^{-1}\left(\frac{b - c\cos(60)}{a}\right)\right)}$$

Note that beta is set to 60 in this equation as we know the person's $\beta$ angle surpasses the limit, being less than 60°. The minimum of $\beta$ is set and will result in the output being the minimum height of the block needed.

## Program/Code

Extending the code in Part 1, to facilitate the application of the formula to obtain the height of the block needed, I continued a program on python, in which the user can type in the lengths of their lower leg, upper leg, and back and the smallest ankle angle they can obtain. The program will then apply the equation for $\beta$ and return the back angle required IF the back angle is over 60°. If the back angle does fit into the restriction, and is under 60°, the program will say so, ask for the length *d* (the heel to the ball of the foot), and will return the height of the block needed for the person in question.

This is the code:

```
from math import acos, cos, sin, pi
import matplotlib.pyplot as pyplot
import numpy as np

def DegreesToRadians(deg):
    return deg*pi/180

def RadiansToDegrees(rad):
    return rad/pi*180

a= float(input("Enter the length of your lower leg, in meters:"))
b= float(input("Enter the length of your upper leg, in meters:"))
c= float(input("Enter your back length from hip to shoulder, in meters:"))
Alphadeg= float(input("Enter the smallest ankle angle you can obtain, in degrees:"))

Alpha= DegreesToRadians(Alphadeg)
Beta= acos((-a*cos(Alpha)+b)/c)
Betadeg= RadiansToDegrees(Beta)

if Betadeg<float(60):
    d= float(input("Enter the distance between your heel and the ball of your foot, in meters:"))
    Gamma= Alpha-acos((b-c*cos(DegreesToRadians(60)))/a) #calculates the necessary block angle
    h= d*sin(Gamma) #converts block angle to block height in meters
    print("You need a block under your heel of ",round(h*100,1)," cm.")

else:
    print("You do not need a block under your heel.")
```

This is a demonstration of the program's "receiver end", in which some potential lengths and ankle flexibility for which the hip angle would be a little under 60° are entered, and the height of the block is successfully returned:

Note that the height is given in cm for convenience, as *h* is multiplied by 100. The height is also rounded to two significant figures as a more detailed number would not be relevant.

```
Enter the length of your lower leg, in meters:0.5
Enter the length of your upper leg, in meters:0.54
Enter your back length from hip to shoulder, in meters:0.47
Enter the smallest ankle angle you can obtain, in degrees:60
Enter the distance between your heel and the ball of your foot, in meters:0.15
You need a block under your heel of  2.0  cm.
```

## The effect of the ankle flexibility on the block height

This graph demonstrates the relationship between the ankle flexibility ($\theta$) on the block height ($h$):

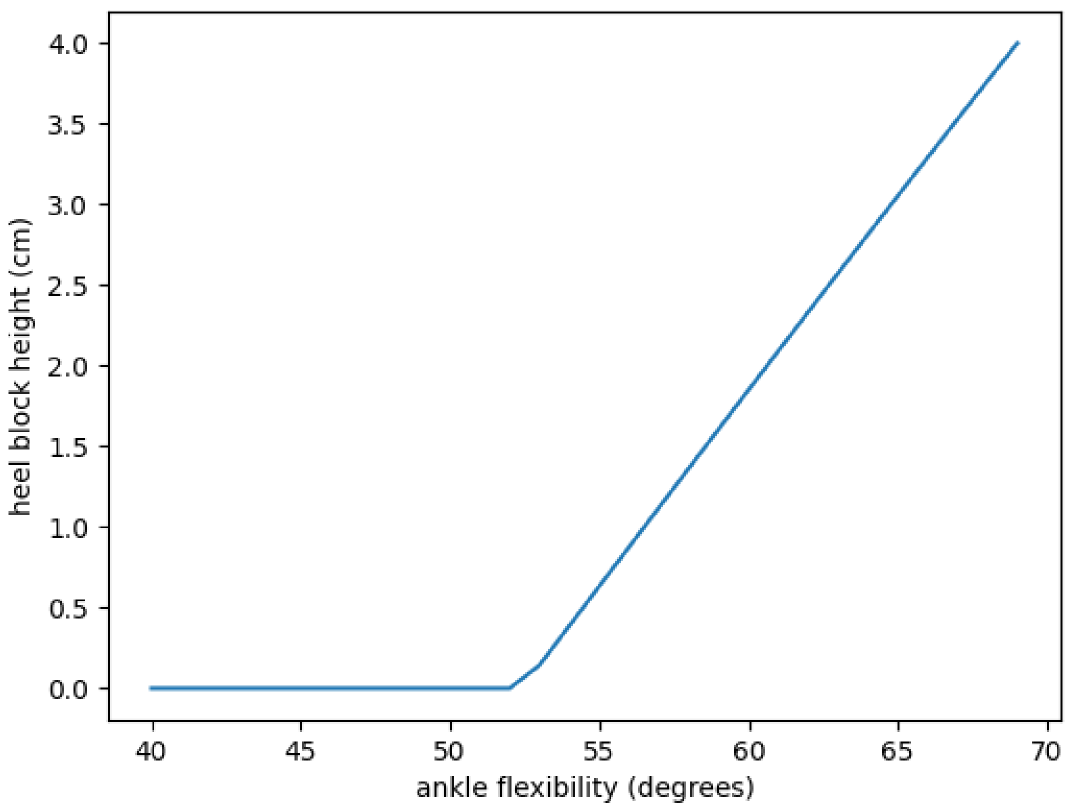



*Figure 13: relationship between ankle flexibility and heel block height.*

This graph is first constant then increasing, and it shows that the bigger the ankle angle (the less ankle flexibility someone has), the bigger the block needed. A flexible ankle would decrease the chance of needing a block and decrease the likely height of it. It can be seen that the point at which the ankle angle leads to needing a block is when the ankle angle is at around 52.5°, with the average human measurements used in Part 1. If that average person has an ankle flexibility above that, a block is needed, whereas if the ankle angle was below it, a block height of 0 (no block) is needed.

# Part 3; Real life application and evaluation

## A real life application

To try the program, I entered an adult female volunteer's measurements:

```
Enter the length of your lower leg, in meters:0.45
Enter the length of your upper leg, in meters:0.46
Enter your back length from hip to shoulder, in meters:0.44
Enter the smallest ankle angle you can obtain, in degrees:65
Enter the distance between your heel and the ball of your foot, in meters:0.15
You need a block under your heel of  1.9  cm.
```

She did a weighted squat without a block, as a reference, on the left. The second picture is using the 2cm block, then 4cm and then then 6cm.

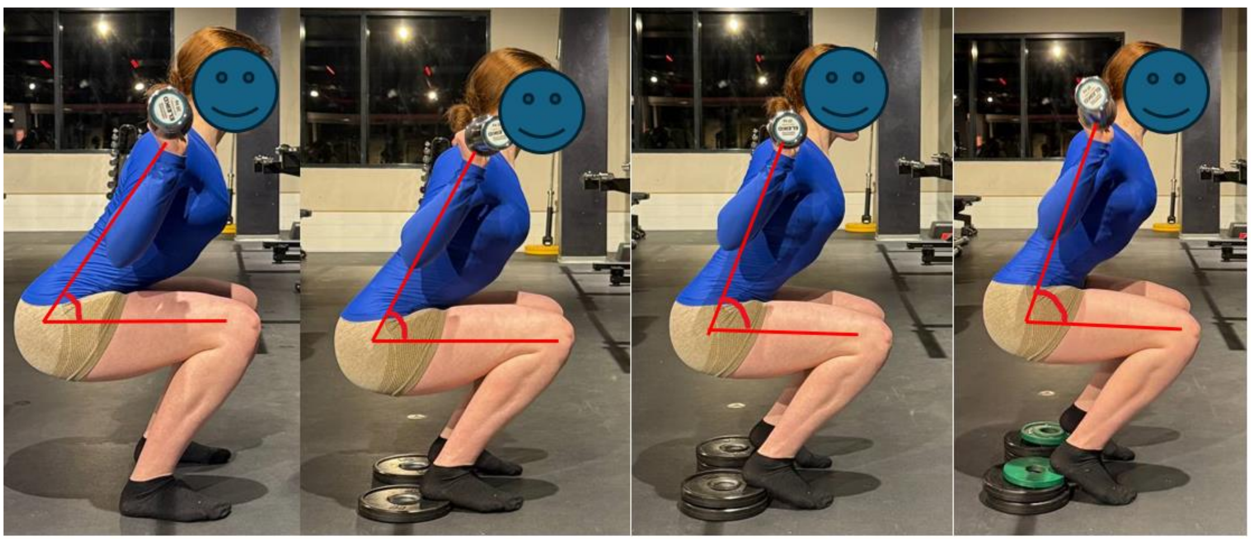


*Figure 14: Real-life application of heel elevation.*

It can be observed in figure 14 that the small of her back is the least straight in the first picture, which is due to my $\beta$ angle being too small. The program tells us that it is mathematically not a good squat either, as the $\beta$ angle is under 60°, so a block is needed. The second picture is much better. Her $\beta$ angle is supposed to be right above 60° if the program is correct, and this seems to be the case. For experimentation purposes, I elevated her heels two steps further to observe the change in the $\beta$ angle, and it is significant. The angle variation between no block and the increasing block sizes can be observed clearly in red in figure 14.

This use of the program and equations ties my investigation together, proving its real-life usefulness and its accuracy.

# Evaluation

The final model made is able to help predict adjustments needed for individuals based on their own individual body dimensions, making this a personalized exploration that is practical and usable by anybody.

## Strengths of the investigation

It is linked to real life and can be useful to anybody attempting this well-known gym exercise.

The program made leads to the formulae being easily applicable.

The equation (and program) was proven useful through putting it to the test in real-life.

## Weaknesses and possible extensions of the investigation

A more accurate investigation could be conducted, as measuring one's joints in the way done in this paper using protractors leads to inaccuracies.

I could have asked a lot of people for their body length data and used that in order to make graphs.

The effectiveness or accuracy of the program could have been tested multiple times using several volunteers of different body measurement ratios.

Someone could have been measured and given a prediction on whether or not they can perform a perfect squat, and they could then test out that hypothesis.

An interesting extension could be the investigation of how these same body proportions affect a similar exercise, the Bulgarian split squat.

# Bibliography

TAYLOR, R. (2020). *11 Weighted Squats Variations You Need to Add to Your Workout Routine*. [online] AQF Sports Official Blog. Available at: https://blog.aqfsports.com/11-weighted-squats-varations/ [Accessed 20 Oct. 2024].

# Appendix

Below is the python code used to generate the graphs in figures 6,7,8 and 13:

```
import matplotlib.pyplot as pyplot
import numpy as np
from math import acos, cos, pi

def DegreesToRadians(deg):
    return deg*pi/180

def RadiansToDegrees(rad):
    return rad/pi*180

a= float(input("Enter the length of your lower leg, in meters:"))
b= float(input("Enter the length of your upper leg, in meters:"))
c= float(input("Enter the length of your back, in meters:"))
Alphadeg= float(input("Enter the smallest ankle angle you can obtain, in degrees:"))
#a=0.56
#b=0.52
#c=0.60
#Alphadeg=65

def Backangle(a,b,c,Alphadeg):
    Alpha= DegreesToRadians(Alphadeg)
    Beta= acos((-a*cos(Alpha)+b)/c)
    return RadiansToDegrees(Beta)

# define empty list of numbers for plotting later:
xs = []
ys = []
# variation in Beta with Alpha:
for x in range(45,75):
    Betadeg= Backangle(a,b,c,x)
    xs.append(x)
    ys.append(Betadeg)

#plot outcome:
fig, ax = pyplot.subplots()
ax.plot(xs, ys)
pyplot.title("relation between ankle and hip angle")
pyplot.xlabel("angle of ankle (degrees)")
pyplot.ylabel("back angle (degrees)")
pyplot.savefig('testplot.png')
pyplot.show()

# reset to empty lists:
xs = []
ys = []
# variation of Beta with a:
for x in range(30,60):
    Betadeg= Backangle(x/100,b,c,Alphadeg)
    xs.append(x)
    ys.append(Betadeg)
fig, ax = pyplot.subplots()
ax.plot(xs, ys)
pyplot.title("relation between lower leg length and hip angle")
pyplot.xlabel("length of lower leg (cm)")
pyplot.ylabel("back angle (degrees)")
pyplot.savefig('lowerleg.png')
pyplot.show()

# reset to empty lists:
xs = []
ys = []
# variation of Beta with b:
```

```
for x in range(45,65):
    Betadeg= Backangle(a,x/100,c,Alphadeg)
    xs.append(x)
    ys.append(Betadeg)
fig, ax = pyplot.subplots()
ax.plot(xs, ys)
pyplot.title("relation between upper leg length and hip angle")
pyplot.xlabel("length of upper leg (cm)")
pyplot.ylabel("back angle (degrees)")
pyplot.savefig('upperleg.png')
pyplot.show()

# reset to empty lists:
xs = []
ys = []
# variation of Beta with c:
for x in range(40,65):
    Betadeg= Backangle(a,b,x/100,Alphadeg)
    xs.append(x)
    ys.append(Betadeg)
fig, ax = pyplot.subplots()
ax.plot(xs, ys)
pyplot.title("relation between back length and hip angle")
pyplot.xlabel("back length (cm)")
pyplot.ylabel("back angle (degrees)")
pyplot.savefig('back.png')
pyplot.show()
```